Microeconomic Theory I — Study Note

Chapter 1: Decision Theory

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1 Decision Theory

Roadmap: Primitive domain → Weak Preference Relation (rationality axioms) → Strict/Indifference derived relations → Utility representation → Choice structures and WARP.

Decision theory provides the axiomatic foundation for individual rational behavior. Before asking how markets aggregate choices, we must understand what it means for a single agent to choose *consistently*. The central objects are a choice set, a preference relation, and a utility function that numerically encodes the preference.

1.1 The Primitive Domain

Definition: Primitive Objects

Three objects define the decision-theoretic primitive:

1. **Choice set** X: the universal set of alternatives available to the agent.
2. **Preference**: a binary comparison over X; given $x, y \in X$, which is weakly better?
3. **Utility**: a function $u : X \to \mathbb{R}$ placing a numerical value on each $x \in X$.

A **weak preference relation** $\succsim$ on X is a subset of $X \times X$ such that $(x, y) \in \succsim$ is written $x \succsim y$ and read "x is at least as good as y." The strict preference $\succ$ and indifference $\sim$ are both *derived* from $\succsim$, not independently assumed:

$$x \sim y \iff x \succsim y \text{ and } y \succsim x, \qquad x \succ y \iff x \succsim y \text{ and } y \not\succsim x.$$

Definition: Rational Preference Relation

A weak preference relation $\succsim$ is *rational* if it satisfies:

- **Completeness**: $\forall\, x, y \in X$, either $x \succsim y$ or $y \succsim x$ (or both).
- **Transitivity**: $\forall\, x, y, z \in X$, $x \succsim y$ and $y \succsim z \Rightarrow x \succsim z$. No cycles.

Rationality implies **reflexivity**: $x \succsim x$ for all $x \in X$.

Limitations: Rationality is the ideal model, but empirically questionable. Small perceptual thresholds may violate transitivity, and majority-rule voting over multi-dimensional alternatives can produce cycles ($a \succ b \succ c$ but $c \succ a$).

■ Transitivity/Rationalizability: Counterexample

Condorcet cycle: $x \succ y \succ z \succ x$.

Proposition: Properties of Derived Relations (Prop. 1.B.1)

Suppose $\succsim$ is rational. Then:

- $\sim$ is reflexive, transitive, and symmetric.

- $\succ$ is irreflexive ($x \not\succ x$) and transitive ($x \succ y$, $y \succsim z \Rightarrow x \succ z$).

Proof

Properties of $\sim$. Reflexivity: $x \succsim x$ (completeness with $x = y$), so $x \sim x$. Symmetry: $x \sim y \Rightarrow x \succsim y$ and $y \succsim x \Rightarrow y \sim x$. Transitivity: $x \sim y$, $y \sim z \Rightarrow x \succsim z$ and $z \succsim x$ by double application of transitivity of $\succsim$.

Irreflexivity of $\succ$. Suppose $x \succ x$: then $x \succsim x$ and $x \not\succsim x$ — contradiction.

Transitivity of $\succ$. Suppose $x \succ y$ and $y \succsim z$. Then $x \succsim z$ by transitivity. If $z \succsim x$, then $z \succsim y$, giving $y \sim z$, contradicting $x \succ y$ (which requires $y \not\succsim x$). ■

1.2 Strict Preference Axiomatization

Definition: Rational Strict Preference Relation (SPR)

One can alternatively take $\succ$ as the primitive. A strict preference relation $\succ$ on X is *rational* if:

1. **Asymmetry**: no x, y such that $x \succ y$ and $y \succ x$.
2. **Negative transitivity**: if $x \succ y$, then for every $z \in X \setminus \{x, y\}$, either $x \succ z$ or $z \succ y$ (or both).

Given a rational SPR $\succ$, recover the weak preference by $x \succsim y \iff \neg(y \succ x)$.

Note: Negative transitivity cannot be replaced by ordinary transitivity alone. With $X = \{x, y, z\}$ and $x \succ y$ only, transitivity holds vacuously but z is unranked; completeness of the induced $\succsim$ is not guaranteed.

■ (PS1: Q1)

- Given rational WPR $\succsim$, $\exists \succ$ rational: define $x \succ y$ as $x \succsim y$ and $y \not\succsim x$.
- Given rational SPR $\succ$, $\exists \succsim$ rational: define $x \succsim y$ as $y \not\succ x$.

1.3 Utility Representation

Definition: Utility Function

A function $u : X \to \mathbb{R}$ *represents* a preference relation $\succsim$ if for all $x, y \in X$:

$$x \succsim y \iff u(x) \geq u(y).$$

Ordinal invariance: if u represents $\succsim$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is strictly increasing, then $\varphi \circ u$ also represents $\succsim$. Utility is unique only up to *strictly increasing (ordinal) transformations.*

Exception: when risk is involved, the curvature of u carries cardinal meaning (risk aversion), so only affine transformations are permitted there.

Theorem: Utility Representation Implies Rationality (Thm. 1.2.1)

If $\succsim$ can be represented by some $u : X \to \mathbb{R}$, then $\succsim$ is rational.

Proof

Let $u : X \to \mathbb{R}$ represent $\succsim$. **Completeness:** for any $x, y \in X$, either $u(x) \geq u(y)$ or $u(y) \geq u(x)$ (real numbers are totally ordered), so either $x \succsim y$ or $y \succsim x$. **Transitivity:** if $x \succsim y$ and $y \succsim z$, then $u(x) \geq u(y) \geq u(z)$, hence $x \succsim z$. ■

Converse is False in General

The converse of Thm. 1.2.1 does not hold in general. The **lexicographic preference** on $\mathbb{R}^2$ ($(x_1, y_1) \succsim (x_2, y_2)$ iff $x_1 > x_2$, or $x_1 = x_2$ and $y_1 \geq y_2$) is rational but admits no utility representation: if u existed, each x would produce a distinct interval $(u(x, \cdot))$ in $\mathbb{R}$ — uncountably many disjoint intervals, a contradiction.

For **finite sets** $|X| < \infty$, rationality *is* sufficient for utility representation.

1.4 Choice Structures and Revealed Preference

Roadmap: Choice structure → Revealed preference → WARP → Rationalizability.

Definition: Choice Structure and Revealed Preference

A *choice structure* $(\mathcal{B}, C(\cdot))$ consists of:

- $\mathcal{B} \subseteq \mathcal{S}(X)$, $\mathcal{B} \neq \varnothing$: a collection of non-empty menus (budget sets the agent may face).
- $C : \mathcal{B} \to \mathcal{B}$: a *choice function* assigning to every $B \in \mathcal{B}$ a non-empty subset $C(B) \subseteq B$ (the chosen elements).

Given $(\mathcal{B}, C(\cdot))$, x is *revealed weakly preferred* to y (written $x \succsim^* y$) if $\exists B \in \mathcal{B}$ such that $x \in C(B)$ and $y \in B$. Write $x \succ^* y$ if additionally $y \notin C(B)$.

Definition: Weak Axiom of Revealed Preference (WARP)

A choice structure $(\mathcal{B}, C(\cdot))$ satisfies *WARP* if: for any $B, B' \in \mathcal{B}$, whenever $x, y \in B \cap B'$, $x \in C(B)$, and $y \in C(B')$, it follows that $x \in C(B')$.

Equivalently: if x is ever chosen when y is available ($x \succsim^* y$), then y should not be chosen when x is available, unless x is also chosen. WARP holds trivially when all menus are binary.

Menu Independence: Adding z to a menu should not change the relative ranking of x and y. If $\{x\} = C(\{x, y\})$, then from $\{x, z, y\}$ we can *never* pick y alone.

■ WARP Counterexample: Context-specific relative flips

Politeness cycle: being polite by not taking the largest piece in the menu: Always pick the second largest item: $C(\{x, y\}) = \{x\}$ and $C(\{x, y, z\}) = \{y\}$ but $x \notin C(\{x, y, z\})$

■ Sen's Axioms (PS1: Q4)

- Sen's α (Downward consistency): If $x \in A \subset B$ and $x \in C(B)$, then $x \in C(A)$: If an item is picked in a larger menu, it should also be picked amongst any subset contained by the menu
- Sen's β (Upward consistency): If $x, y \in C(A)$ and $A \subset B$, if $y \in C(B)$, then $x \in C(B)$. If both items are picked in a smaller menu, then one cannot uniquely beat the other in when expanding the menu.

Remark: Sen's $\alpha + \beta \Leftrightarrow$ WARP

Definition: Preference-Generated Choice

Given a rational $\succsim$ on X and $\mathcal{B} \subseteq \mathcal{S}(X)$:

$$C^*(B, \succsim) \;=\; \{x \in B : x \succsim y \text{ for all } y \in B\}.$$

One can show $C^*(B, \succsim) \neq \varnothing$ whenever $\succsim$ is rational and B is finite.

Theorem: WARP from Rationality (Thm. 1.2.2)

If $\succsim$ is rational, then $(\mathcal{B}, C^*(\cdot, \succsim))$ satisfies WARP.

Proof

Suppose $x, y \in B \cap B'$, $x \in C^*(B, \succsim)$, and $y \in C^*(B', \succsim)$. Then $x \succsim z$ for all $z \in B$ (in particular $x \succsim y$), and $y \succsim z$ for all $z \in B'$. For any $z \in B'$: $y \succsim z$ and $x \succsim y$; by transitivity $x \succsim z$. Hence $x \in C^*(B', \succsim)$. ■

Proposition: Rationalizability from WARP (Prop. 1.2.2)

Suppose $C(\cdot)$ satisfies WARP and $\mathcal{B} = \mathcal{S}(X)$ with $|X| \leq 3$. Then there exists a unique rational $\succsim$ such that $C(\cdot) = C^*(\cdot, \succsim)$.

Proof sketch: Define $x \succsim y$ iff $x \succsim^* y$. WARP on binary menus establishes completeness and, together with menus of size 3, establishes transitivity. With $|X| \leq 3$ all cycles can be ruled out directly. Uniqueness: binary choices pin down every pairwise comparison. ■

Three-Step Methodology

1. Give the agent choices to observe trade-offs.
2. Use choices to infer preferences (revealed preference $\succsim^*$).
3. Use preferences to find a utility representation.

WARP is the minimal discipline ensuring this cycle is consistent. Without it, revealed preferences can cycle: $x \succ^* y \succ^* z \succ^* x$.

1.5 Summary

Choice structure (Behavioral) $\Rightarrow$ Ordering

- **WARP** is the behavioral rule: If x beats y once both are available, under no circumstance would y uniquely beat x
- **WARP** on **all finite menus** implies we have a **rational** (complete and transitive) preference relation $\succsim$

* **WARP** on restricted domain is NOT enough for rational $\succsim$ as cycles can be hidden in the menus you don't observe.