Microeconomic Theory I: Study Note

Quarter 1: Preference, Choice, and Utility

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3 Consumer Choices

Roadmap: Consumption set and budget set → Walrasian demand (homogeneity, Walras' Law) → Comparative statics (Euler, Cournot, Engel aggregation) → WARP budget version → Law of Compensated Demand → Slutsky matrix.

3.1 Budget Set

Definition: Consumption Set and Walrasian Budget Set

The *consumption set* is $X = \mathbb{R}^L_+$; elements $x \in X$ are *commodity bundles.*
Given price vector $p \in \mathbb{R}^L_{++}$ and wealth $w > 0$, the *Walrasian budget set* is

$$B(p, w) := \{x \in \mathbb{R}^L_+ : p \cdot x \leq w\}.$$

$B(p, w)$ is **convex**: for $x, x' \in B(p, w)$ and $\lambda \in [0, 1]$, $p\cdot(\lambda x+(1-\lambda)x') = \lambda(p\cdot x)+(1-\lambda)(p\cdot x') \leq w$.

Geometry: the normal to the budget hyperplane is p; any two points $x, \widetilde{x}$ on the boundary satisfy $p(x - \widetilde{x}) = 0$. Increasing w shifts the hyperplane outward parallel to itself (leaving p unchanged). If $p_i = 1$ (good i is numéraire), then $w = x_i$ for the bundle concentrated on good i.

3.2 Walrasian (Marshallian) Demand

Definition: Walrasian Demand and Its Properties

The *Walrasian demand correspondence* $x(p, w)$ assigns to each (p, w) the chosen consumption vector(s). It is a function when preferences are strictly convex.

- **Homogeneity of degree 0:** $x(tp, tw) = x(p, w)$ for all $t > 0$. Follows from $B(tp, tw) = B(p, w)$: scaling all prices and wealth proportionally leaves the budget set unchanged. No nominal illusion.
- **Walras' Law:** $p \cdot x(p, w) = w$ for all p, w. The agent exhausts the entire budget. [a] A locally non-satiated preference always implies Walras' Law.

[a] Fails if X is discrete (agent may not exactly exhaust w).

3.3 Comparative Statics and Aggregation

Definition: Good Types by Wealth Response

Fix p. By sign of $\partial x_i/\partial w$:

- **Normal good**: $\partial x_i/\partial w > 0$ more when richer.
- **Inferior good**: $\partial x_i/\partial w < 0$ less when richer.

- **Ordinary good**: $\partial x_i/\partial p_i < 0$ (own-price demand slopes down).
- **Giffen good**: $\partial x_i/\partial p_i > 0$ (own-price demand slopes up).

The *Engel function* is $w \mapsto x(\overline{p}, w)$ at fixed $\overline{p}$; its image in commodity space is the *wealth expansion path*.

Proposition: Three Aggregation Relations

Differentiating the two maintained assumptions yields:

Euler's relation (from homogeneity, differentiate w.r.t. t at $t = 1$):

$$D_p x(p, w) \cdot p + D_w x(p, w) \cdot w = 0. \qquad \text{Elasticity form: } \sum_j \varepsilon_{ij} + \varepsilon_{iw} = 0.$$

Cournot aggregation (differentiate Walras' Law w.r.t. p):

$$p \cdot D_p x(p, w) + x(p, w)^\top = 0.$$

Engel aggregation (differentiate Walras' Law w.r.t. w):

$$p \cdot D_w x(p, w) = 1.$$

3.4 WARP: Budget Version and the Slutsky Matrix

Definition: WARP (Budget Version)

The Walrasian demand $x(p, w)$ satisfies *WARP* if: whenever $p \cdot x(p', w') \leq w$ and $x(p, w) \neq x(p', w')$, then $p' \cdot x(p, w) > w'$.

In this case, $x(p, w) R^D x(p', w')$ as $x(p, w)$ is directly revealed preferred to $x(p', w')$

Interpretation: if $x(p', w')$ was affordable under (p, w) but not chosen, then $x(p, w)$ must be *not affordable* under (p', w'). Price and compensated demand changes move in opposite directions.

Proposition: Law of Compensated Demand (LCD)

$x(p, w)$ satisfies WARP if and only if, for any compensated price change ($w' = p' \cdot x(p, w)$):

$$(p' - p) \cdot (x(p', w') - x(p, w)) \ \leq \ 0.$$

in differential form:

$$dp \cdot dh \leq 0$$

and we decompose dh into $\partial h \cdot dp$ which $\partial h = S$, so to get the following sandwich form $dpSdp$

Proof

Suppose $w' = p' \cdot x(p,w)$ (compensated). Then $x(p,w) \in B(p',w')$. If $x(p,w) \neq x(p',w')$, WARP gives $p \cdot x(p',w') > w$. Then:

$$(p' - p) \cdot (x' - x) = p' \cdot x' - p' \cdot x - p \cdot x' + p \cdot x = w' - w' - p \cdot x' + w = w - p \cdot x' \leq 0,$$

since $p \cdot x' > w$ by WARP. ■
In differential form: $dp \cdot [D_p x \cdot dp + D_w x \cdot dw] \leq 0$ under $dw = x \cdot dp$, which is $dp^\top S(p,w)\, dp \leq 0$, i.e. the **Slutsky matrix** S is negative semidefinite.

Definition: Slutsky Matrix

The *Slutsky matrix* $S(p,w)$ has (i,j)-entry:

$$S_{ij}(p,w) := \frac{\partial h_i(p,u)}{\partial p_j} = \frac{\partial x_i}{\partial p_j}(p,w) + \frac{\partial x_i}{\partial w}(p,w) \cdot x_j(p,w).$$

Slutsky element: (S.E. + I.E.) - I.E.
Properties under WARP:

1. **Negative semidefinite**: $v^\top S v \leq 0$ for all $v \in \mathbb{R}^L$ (from LCD).[a]
2. $S \cdot p = 0$: Walras' Law implies $p \cdot S = 0$ and $S \cdot p = 0$.
3. **Homogeneity of degree 0**: consistent with $x(p,w)$ hod 0.
4. **Symmetry**[b]: $S_{ij} = S_{ji}$ — requires integrability, holds when demand is generated by utility maximization.

Giffen goods: $\partial x_i / \partial p_i > 0$ requires $\partial x_i / \partial w < 0$ (must be inferior). Since $S_{ii} \leq 0$ (NSD diagonal): $\partial x_i / \partial p_i = S_{ii} - (\partial x_i / \partial w) x_i > 0$ only if the income effect dominates.

[a] Meaning $\sum_l dp_l dh_l \leq 0$: price and quantity moves opposite on average. Once income effect is neutralized, we cannot have a Giffen good.
[b] Only guaranteed under $L = 2$

WARP Numerical Example ($L = 3$, $w = 8$)

$(p^1, x^1) = ((2,2,1),(1,2,2))$: $p^1 \cdot x^1 = 2 + 4 + 2 = 8$, affordable $(p^2, x^2) = ((1,2,2),(2,1,2))$: $p^2 \cdot x^2 = 2 + 2 + 4 = 8$, affordable
check $p^1 \cdot x^2 = 4 + 2 + 2 = 8$, so x^2 is affordable under p^1. Then by WARP it has to be that x^1 not affordable under p^2. We check it by: $p^2 \cdot x^1 = 1 + 4 + 4 = 9 > 8$. WARP satisfied!
x^1 was affordable under p^2 but not chosen, and x^2 is not affordable under p^1. ✓.