Microeconomic Theory I — Study Note

Chapter 4: Utility Maximization

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4 Utility Maximization

Roadmap: Representation (continuity, convexity) → UMP and Lagrangian → Indirect utility → Duality and EMP → Hicksian demand → Slutsky equation → Roy's identity → GARP and Afriat → Welfare (EV, CV).

4.1 Representation Theory

Definition: Continuity and Debreu's Theorem

A preference relation $\succsim$ is *continuous* if for sequences $x_n \to x$ and $y_n \to y$ with $x_n \succsim y_n$ for all n, we have $x \succsim y$. Equivalently, upper and lower contour sets are closed.

Failure case: lexicographic preference on $\mathbb{R}^2$ is rational but *not* continuous (upper contour set $\{y : y \succsim x\}$ is not closed), hence admits no utility representation.

Theorem: Debreu's Representation Theorem

A preference relation $\succsim$ on $\mathbb{R}^L_+$ admits a *continuous* utility function representation if and only if $\succsim$ is rational and continuous.

Definition: Convexity

- $\succsim$ on $\mathbb{R}^L_+$ is *convex* if $x \succsim y$ and $x' \succsim y$ imply $\alpha x + (1-\alpha)x' \succsim y$ for all $\alpha \in [0,1]$. Strictly convex if the implication is strict for $x \neq x'$.
- $u : X \to \mathbb{R}$ is *convex* if $u(\alpha x + (1-\alpha)x') \geq \alpha u(x) + (1-\alpha)u(x)$. [a]

[a] convexity is a cardinal property that is not necessarily invariant under monotone transformation

Definition: Quasiconcavity

$u : X \to \mathbb{R}$ is *quasiconcave* if upper contour sets $\{x : u(x) \geq \overline{u}\}$ are convex, equivalently $f(\alpha x + (1-\alpha)y) \geq \min\{f(x), f(y)\}$ for all $\alpha \in [0,1]$. [a]

[a] quasiconcavity is *ordinal* (preserved under monotone transformations of u), unlike concavity.

Argmax sets: quasiconcavity ⇒ argmax set is convex; strict quasiconcavity ⇒ unique argmax.

4.2 The Utility Maximization Problem

Definition: UMP and First-Order Conditions

The consumer solves:

$$\max_{x \in \mathbb{R}^L_+} u(x) \quad \text{s.t.} \quad p \cdot x \leq w.$$

Lagrangian: $\mathcal{L} = u(x) - \lambda(p \cdot x - w)$, $\lambda \geq 0$. First-order conditions: $u_{x_i} = \lambda p_i$ for all i, so:

$$\frac{u_{x_i}}{u_{x_j}} = \frac{p_i}{p_j} \quad \forall i,j \qquad \text{(MRS = slope of budget line).}$$

Local non-satiation (LNS): for every x and $\varepsilon > 0$, $\exists\, y$ with $d(x, y) < \varepsilon$ and $y \succ x$. LNS implies Walras' Law: if $p \cdot x(p, w) < w$, an ε-ball around $x(p, w)$ lies inside $B(p, w)$; by LNS find $y \succ x(p, w)$ in the ball, contradicting optimality.

Definition: Indirect Utility Function and Shadow Price

$$v(p, w) := \max_x u(x) \quad \text{s.t.} \quad p \cdot x \le w.$$

The Lagrange multiplier λ has the interpretation $\lambda = \partial v/\partial w$: the *marginal value of wealth* (shadow price of the budget constraint). Key: $\lambda \ge 0$, and $V(z) := v$ with constraint relaxed to $p \cdot x \le w + z$ is increasing and concave in z.

Definition: Roy's identity:

Indirect utility $v(p, w) \Rightarrow$ Marshallian demand $x_l(p, w)$.

$$x_l(p, w) = -\frac{v_{p_l}}{v_w}$$

From Lagrangian:

$$\mathcal{L}(x, \lambda; p, w) = u(x) - \lambda(p \cdot x - w)$$

evaluated at $v(p, w) = u(x(p, w))$:

$$v(p, w) = u(x(p, w)) - \lambda \left(\sum_{k=1}^{L} p_l x_l(p, w) - w \right)$$

We have the optimization result that $\frac{\partial \mathcal{L}}{\partial x_l} = 0$, $\frac{\partial \mathcal{L}}{\partial \lambda} = 0$. We take total differential for v w.r.t. p_l and w:

$$\begin{aligned}
\frac{\partial v}{\partial p_l} &= \frac{\partial \mathcal{L}}{\partial p_l} + \sum_{k=1}^{L} \underbrace{\frac{\partial \mathcal{L}}{\partial x_k}}_{=0} \frac{\partial x_k}{\partial p_l} \\
&= \frac{\partial \mathcal{L}}{\partial p_l} \quad \text{else fixed} \\
&= -\lambda x_l(p, w) \\
\frac{\partial v}{\partial w} &= \frac{\partial \mathcal{L}}{\partial w} + \sum_{k=1}^{L} \underbrace{\frac{\partial \mathcal{L}}{\partial x_k}}_{=0} \frac{\partial x_k}{\partial p_l} + \underbrace{\frac{\partial \mathcal{L}}{\partial \lambda}}_{=0} \frac{\partial \lambda}{\partial w} \\
&= \frac{\partial \mathcal{L}}{\partial w} \\
&= \lambda
\end{aligned}$$

We magically take the ratio to cancel out the multiplier:

$$-\text{Ratio} = -\frac{v_{p_l}}{v_w} = x_l(p, w)$$

4.3 Duality: EMP and Hicksian Demand

Definition: Expenditure Minimization Problem (EMP)

The *EMP* is the dual to the UMP:

$$e(p, \overline{u}) := \min_x p \cdot x \quad \text{s.t.} \quad u(x) \geq \overline{u}.$$

Properties:

1. homogeneous of degree 1 in p;
2. strictly increasing in $\overline{u}$ (under LNS);
3. non-decreasing in each p_ℓ;
4. **concave in** p (key: $D_p^2 e$ is NSD and symmetric by Young's theorem).

Duality identities:

- Expenditure $e \Rightarrow$ IU $v \Rightarrow$ utility: $u = v(p, e(p, u))$
- IU $v \Rightarrow$ expenditure $e \Rightarrow$ income: $w = e(p, v(p, w))$

The two give us the same optimizer bundles: $h(p^*, u^*) = x(p^*, w^*)$.

Definition: Hicksian Demand

The *Hicksian (compensated) demand* $h(p, \overline{u}) = \arg\min e(p, \overline{u})$.

- homogeneous of degree 0 in p
- convex-valued (quasi-concave u)
- unique (strictly quasi-concave u)

λ Algebraic Trick

When backing out Hicksian demand, it is convenient to solve for the multiplier first.

Theorem: Shephard's Lemma

Expenditure function $e \Rightarrow$ Hicksian demand h_l

For u continuous, LNS, with strictly convex $\succsim$:

$$h(p, \overline{u}) = \nabla_p e(p, \overline{u}).$$

Proof: $e(p,\overline{u}) = \inf_{x\in K} p \cdot x$ is the support function of the convex upper contour set $K = \{x : u(x) \geq \overline{u}\}$. For convex K, $\nabla_p \mu_K(p) = \overline{x}_p$ (gradient of support function recovers the supporting point). ■

4.4 Slutsky Equation

Theorem: Slutsky Equation

By duality $h(p,\overline{u}) = x(p, e(p,\overline{u}))$. Differentiating w.r.t. p_k and reorganize:

$$\underbrace{\frac{\partial x_\ell}{\partial p_k}}_{\text{Total change}} = \underbrace{\frac{\partial h_\ell}{\partial p_k}}_{\text{S.E.}} + \underbrace{\left(-\frac{\partial x_\ell}{\partial w} \cdot x_k\right)}_{\text{I.E.}}$$

In matrix form: $D_p h(p,\overline{u}) = D_p x(p,w) + D_w x(p,w) \cdot x(p,w)^\top =: S(p,w)$.
The Slutsky matrix equals the matrix of Hicksian demand derivatives: NSD and symmetric (by Shephard's lemma and Young's theorem).

To show this,

$$\frac{\partial h_\ell(p,u)}{\partial p_k} = \frac{\partial x_\ell(p,w)}{\partial p_k}\bigg|_{w=e(p,u)} + \frac{\partial x_\ell(p,w)}{\partial w}\bigg|_{w=e(p,u)} \cdot \underbrace{\frac{\partial e(p,u)}{\partial p_k}}_{=h_l \text{ Shepard's}}$$

$\frac{\partial e(p,u)}{\partial p_k} = h_l(p,u)$ by Shepard's lemma, and at optimum equals to the Marshallian demand $x_l(p, e(p,u))$.

▶ ***Summary :UMP–EMP Duality Diagram***

Four linked objects:

- $v(p,w) \xrightarrow{\text{Roy}} x(p,w)$: Walrasian demand from indirect utility.
- $e(p,\overline{u}) \xrightarrow{\text{Shephard}} h(p,\overline{u})$: Hicksian demand from expenditure.
- $\nabla_p x(p,w) \xrightarrow{\text{Slutsky}} \nabla_p h(p,\overline{u})$: substitution + income decomposition.
- Duality identity: $h(p,u) = x(p, e(p,u))$ at optimum
- Closing identities: $u = v(p, e(p,u))$ and $w = e(p, v(p,w))$.

4.5 GARP, Afriat, and Welfare

Definition: GARP & SARP

GARP (Generalized Axiom of Revealed Preference): given data $\{x_i, p_i\}_{i=1}^n$, whenever $x_1 \succsim^* \cdots \succsim^* x_n$ (a revealed preference chain), we require $p_n \cdot (x_1 - x_n) \geq 0$.
It is SARP if we require $p_n \cdot (x_1 - x_n) > 0$.
GARP $\Rightarrow$ WARP, but not conversely.

Definition: Afriat's Theorem

GARP $\iff$ there exists a utility function (piecewise linear, continuous, strictly monotone, concave) that generates the data.

Definition: Equivalent Variation (EV) and Compensating Variation (CV)

Given a price change $p^0 \to p^1$ with $u^0 = v(p^0, w)$ and $u^1 = v(p^1, w)$:

- EV: Old price, New utility[a]

$$EV := e(p^0, u^1) - e(p^0, u^0) = e(p^0, u^1) - w = \int_{p^0}^{p^1} h(p, u^1)\, dp.$$

- CV: New price, Old utility[b]

$$CV := e(p^1, u^0) - e(p^1, u^1) = w - e(p^1, u^0) = \int_{p^1}^{p^0} h(p, u^0)\, dp.$$

EV vs. CV and good type: $EV > CV \iff x$ is a normal good; $EV < CV \iff x$ is an inferior good.
Proof sketch: $EV - CV = \int_{p^0}^{p^1} [h(p, u^1) - h(p, u^0)]\, dp$. For a price decrease ($dp_i < 0$) and normal good ($h(p, u^1) > h(p, u^0)$, since higher utility raises Hicksian demand for normal goods), the integrand is positive and $dp_i < 0$ gives $EV > CV$.

[a] Instead of changing price, we measure the **equivalent** amount of cash to confiscate from consumers.
[b] Amount of cash **Compensated** to consumers to go back to old utility

■ EV and CV

Their signs are the same as welfare change. For non-Giffen goods, $p_k \uparrow$ implies $EV, CV \downarrow$.

Cobb–Douglas UMP

$\max_x \alpha \log x_1 + (1-\alpha) \log x_2$ s.t. $p_1 x_1 + p_2 x_2 \leq w$. FOC: $\alpha / x_1 = \lambda p_1$, $(1-\alpha)/x_2 = \lambda p_2$, so $x_1^* = \alpha w / p_1$, $x_2^* = (1-\alpha) w / p_2$, $\lambda^* = 1/w$.
Indirect utility: $v(p, w) = \alpha \log(\alpha w / p_1) + (1-\alpha) \log((1-\alpha) w / p_2)$. Verify Roy: $-(\partial v / \partial p_1)/(\partial v / \partial w) = \alpha w / p_1 = x_1^*$. ✓