Microeconomic Theory I — Study Note

Chapter 5: Monotone Comparative Statics

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5 Monotone Comparative Statics

Roadmap: Classical IFT approach → Increasing differences → MCS theorem → Supermodularity → Single crossing.

The classical approach — differentiate the FOC and apply the implicit function theorem — requires differentiability and a unique interior solution. Monotone Comparative Statics (MCS) provides a sharper, more general toolkit: under *increasing differences* or *single crossing*, the optimizer moves monotonically with the parameter, without any smoothness assumptions.

5.1 Classical Approach via the Implicit Function Theorem

Definition: Parametric Optimization and Classical Comparative Statics

$\max_{x \in S} f(x, \theta)$. Assuming $f \in \mathcal{C}^2$ and unique interior x^*:

1. **FOC**: $f_x(x^*, \theta) = 0$.
2. **Implicit function**: write $x^* = x(\theta)$, so $f_x(x(\theta), \theta) = 0$.
3. **Differentiate**: $f_{x\theta} + f_{xx} \cdot x'(\theta) = 0$.
4. **Solve**: $\boxed{x'(\theta) = -f_{x\theta}/f_{xx}.}$

Uniqueness (f concave $\Rightarrow f_{xx} < 0$) gives $x'(\theta) \propto f_{x\theta}$: the sign of the comparative static equals the sign of the *cross partial*, also called the *complementarity* of x and θ.

Becker Fertility Model

$u(c, k)$ over consumption c and children k. Budget: $wT + y \geq c + (p + tw)k$ (each child costs p money and t time valued at wage w). Substituting the binding budget:

$$U(k) := u(wT + y - (p + tw)k,\ k).$$

FOC: $U_k = 0 \Rightarrow -(p + tw)u_c + u_k = 0$.
Effect of wage on k^*: $k'(w) \propto -tu_c - (p+tw)(T-tk)u_{cc} + (T-tk)u_{kc}$. The first term $-tu_c < 0$ (crowding out: higher wages raise the time cost of children). The sign of $k'(w)$ depends on the cross-partial u_{kc} — this ambiguity motivates the MCS approach.

5.2 Increasing Differences and MCS

Definition: Increasing Differences

$f : X \times \Theta \to \mathbb{R}$ has *increasing differences* in (x, θ) if for all $x_1 > x_2$ and $\theta_1 > \theta_2$:

$$f(x_1, \theta_1) - f(x_1, \theta_2) \ \geq\ f(x_2, \theta_1) - f(x_2, \theta_2).$$

Strictly increasing differences if the inequality is strict.

Equivalent for $\mathcal{C}^2$ functions: I.D. $\Leftrightarrow f_{x\theta} \geq 0$; strict I.D. $\Leftrightarrow f_{x\theta} > 0$.

Key: no differentiability required in the definition

It is a **cardinal** property.

Theorem: Monotone Comparative Statics (MCS)

Let $\phi(\theta) := \arg\max_{x \in S} f(x, \theta)$. If f has *strictly increasing differences* in (x, θ), then for $\theta_1 > \theta_2$, $x_1 \in \phi(\theta_1)$, $x_2 \in \phi(\theta_2)$:

$$x_1 \;\geq\; x_2.$$

If choice variable and parameter exhibits increasing difference (they like each other), then the optimizer is non-decreasing in θ.

Proof

Suppose for contradiction $x_1 < x_2$. Since $x_1 \in \phi(\theta_1)$: $f(x_1, \theta_1) \geq f(x_2, \theta_1)$, i.e. $f(x_2, \theta_1) - f(x_1, \theta_1) \leq 0$.

Since $x_2 \in \phi(\theta_2)$: $f(x_2, \theta_2) \geq f(x_1, \theta_2)$, i.e. $f(x_2, \theta_2) - f(x_1, \theta_2) \geq 0$.

But strictly increasing differences with $x_2 > x_1$ and $\theta_1 > \theta_2$ gives:

$$f(x_2, \theta_1) - f(x_1, \theta_1) > f(x_2, \theta_2) - f(x_1, \theta_2) \geq 0,$$

contradicting the first inequality. ■

5.3 Single Crossing

Definition: Strict Single Crossing

$F(x; t)$ satisfies *strict single crossing* if whenever $x > x'$ and $\theta_1 > \theta_2$:

$$f(x; \theta_2) \;\geq\; f(x'; \theta_2) \implies f(x; \theta_1) \;>\; F(x'; \theta_1).$$

If x beats x' at a lower parameter θ_2, it strictly beats x' at any higher parameter.

It is an **ordinal** property.

Let $\Delta(\theta) = f(x, \theta) - f(x', \theta)$ be the payoff difference between a high and low action under a given θ. As θ grows, $\Delta(\theta)$ can only cross the zero line only once from negative to positive.

Theorem: Single Crossing implies MCS

Strict single crossing $\Rightarrow$ Monotone Comparative Statics.

Proof

Let $\theta_1 > \theta_2$, $x_2 \in \phi(\theta_2)$, and suppose $x_1 < x_2$ for some $x_1 \in \phi(\theta_1)$. Since $x_2 \in \phi(\theta_2)$: $F(x_2; \theta_2) \geq F(x_1; \theta_2)$. By strict single crossing with $x_2 > x_1$ and $\theta_1 > \theta_2$: $F(x_2; \theta_1) > F(x_1; \theta_1)$, contradicting $x_1 \in \phi(\theta_1)$. ■

I.D. vs. Single Crossing

Single crossing is strictly *weaker* than increasing differences:

- I.D. is a **cardinal** condition: not preserved under monotone transformations of f.
- Single crossing is **ordinal**: preserved under monotone transformations. Only requires that the ranking of x vs. x' crosses at most once as t increases.