Microeconomic Theory I — Study Note

Chapter 7: Risk Preferences

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7 Risk Preferences

Roadmap: Lotteries as CDFs → Risk aversion and Jensen's inequality → Certainty equivalence → Probability premium → Arrow–Pratt measure → TFAE → Comparative risk aversion.

We specialize the EU framework to lotteries over money. The curvature of the Bernoulli utility $u : \$ \to \mathbb{R}$ has a precise behavioral interpretation: concavity ↔ risk aversion. This cardinal content distinguishes risk preferences from the purely ordinal preferences of earlier chapters.

7.1 Risk Aversion and Jensen's Inequality

Definition: Setting: Lotteries over Money

A lottery over money is defined by a CDF $F : \mathbb{R} \to [0, 1]$. The vNM expected utility of F given Bernoulli $u : \mathbb{R} \to \mathbb{R}$ (increasing, continuous) is:

$$U(F) := \int u(x)\,\mathrm{d}F(x) = \mathbb{E}[u(X)].$$

Definition: Risk Aversion, Risk Loving, Risk Neutrality

Risk averse: $\mathbb{E}[u(X)] \le u(\mathbb{E}[X])$ for all $F \Leftrightarrow u$ is **concave**.
Risk loving: $\mathbb{E}[u(X)] \ge u(\mathbb{E}[X])$ for all $F \Leftrightarrow u$ is **convex**.
Risk neutral: $\mathbb{E}[u(X)] = u(\mathbb{E}[X]) \Leftrightarrow u$ is **linear**.

Proposition: Jensen's Inequality

For any lottery F and Bernoulli u: u concave $\Rightarrow \mathbb{E}[u(X)] \le u(\mathbb{E}[X])$.
Proof: For concave u, $\exists\,\lambda$ s.t. $u(x) \le u(\mu) + \lambda(x - \mu)$ for all x (supporting hyperplane at $\mu = \mathbb{E}[X]$). Integrating: $\int u\,\mathrm{d}F \le u(\mu) + \lambda \int (x - \mu)\,\mathrm{d}F = u(\mu)$. ■

▶ *Concavity and Risk Aversion*

For concave u: the chord between $(x - \varepsilon, u(x - \varepsilon))$ and $(x + \varepsilon, u(x + \varepsilon))$ lies *below* the curve. The expected utility $\mathbb{E}[u(X)]$ (a point on the chord) is less than $u(\mathbb{E}[X])$ (the value on the curve at the mean). The gap between them is the risk premium.

7.2 Certainty Equivalent and Probability Premium

Definition: Certainty Equivalent

The *certainty equivalent* $C(F, u) \in \mathbb{R}$ satisfies $u(C(F, u)) = \mathbb{E}[u(X)]$. It is the certain amount the agent finds indifferent to the gamble F.
For risk-averse u: $C(F, u) \le \mathbb{E}[X]$ for all F. The three equivalent conditions chain as:

$$C(F, u) \le \int x\,\mathrm{d}F, \qquad u(C(F, u)) \le u\left(\int x\,\mathrm{d}F\right), \qquad \int u\,\mathrm{d}F \le u\left(\int x\,\mathrm{d}F\right).$$

Definition: Probability Premium

Given wealth x, spread $\varepsilon > 0$, the *probability premium* $\pi(x, \varepsilon, u)$ is the excess probability on the good outcome $x + \varepsilon$ that makes the agent indifferent to bearing symmetric risk:

$$u(x) = \left(\tfrac{1}{2} + \pi\right) u(x + \varepsilon) + \left(\tfrac{1}{2} - \pi\right) u(x - \varepsilon).$$

$\pi > 0$ for a risk-averse agent (requires being "paid" extra probability on the upside to accept the gamble).

7.3 Arrow–Pratt Measure

Definition: Arrow–Pratt Coefficient of Absolute Risk Aversion

$$r_A(x) := -\frac{u''(x)}{u'(x)}.$$

For concave u: $u'' < 0$ and $u' > 0$, so $r_A(x) > 0$. Captures the rate at which marginal utility diminishes.

Proposition: Arrow–Pratt from the Probability Premium

$$r_A(x) = \lim_{\varepsilon \to 0} 4\pi(x, \varepsilon, u).$$

Proof: Taylor expand $u(x \pm \varepsilon)$ around x. Substituting into the probability premium equation and canceling $u(x)$: $0 = 2\pi\varepsilon u'(x) + \frac{\varepsilon^2}{2} u''(x) + O(\varepsilon^3)$. Solving: $\pi = \frac{\varepsilon}{4} r_A(x) + O(\varepsilon^2)$, giving $r_A(x) \approx 4\pi/\varepsilon$. ■

7.4 TFAE and Comparative Risk Aversion

Theorem: Equivalences for Risk Aversion

The following are equivalent for a DM with Bernoulli u:

1. The DM is risk averse.
2. u is concave.
3. $\pi(x, \varepsilon, u) \geq 0$ for all $x, \varepsilon > 0$.
4. $C(F, u) \leq \mathbb{E}[X]$ for all $F \in \mathcal{L}$.

Theorem: Comparative Risk Aversion

The following are equivalent ("u_2 is more risk averse than u_1"):

1. $r_A(x, u_2) \geq r_A(x, u_1)$ for all x.
2. $\exists$ increasing concave ψ s.t. $u_2 = \psi(u_1)$.

3. $C(F, u_2) \leq C(F, u_1)$ for all F.

4. $\pi(x, \varepsilon, u_2) \geq \pi(x, \varepsilon, u_1)$ for all x, ε.

5. Whenever u_2 accepts a lottery over riskless $\overline{x}$, so does u_1.

Proof of (1) $\Leftrightarrow$ (2)

(2) $\Rightarrow$ (1): $u_2 = \psi(u_1)$ with ψ concave. Then $u_2' = \psi'(u_1)u_1'$ and $u_2'' = \psi''(u_1)(u_1')^2 + \psi'(u_1)u_1''$. So:

$$r_A(x, u_2) = -\frac{u_2''}{u_2'} = r_A(x, u_1) - \frac{\psi''(u_1)}{\psi'(u_1)}u_1' \geq r_A(x, u_1),$$

since $\psi'' \leq 0$ and $\psi', u_1' > 0$.
(1) $\Rightarrow$ (2): Define $\psi := u_2 \circ u_1^{-1}$; then $u_2 = \psi(u_1)$. $r_A(u_2) \geq r_A(u_1)$ implies $-\psi''/\psi' \geq 0$, i.e. $\psi'' \leq 0$. ■