Microeconomic Theory I — Study Note

Chapter 8: Stochastic Dominance

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8 Stochastic Dominance

Roadmap: FOSD → SOSD and mean-preserving spreads → Value of information → Garbling and Blackwell ordering.

Stochastic dominance provides criteria for ranking lotteries robust to partial knowledge of u. Rather than fixing u and comparing $\mathbb{E}_F[u]$ vs. $\mathbb{E}_G[u]$, we ask: for which *classes* of u does one lottery always dominate another?

8.1 First-Order Stochastic Dominance

Definition: FOSD

F *first-order stochastically dominates* G (written $F \succsim_1 G$) if for every non-decreasing u:

$$\int u \, \mathrm{d}F \;\geq\; \int u \, \mathrm{d}G.$$

Proposition: FOSD Characterization

$F \succsim_1 G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The CDF of F lies *below* G everywhere (F is "shifted right" relative to G).

Proof (⇒): Fix t, let $u = \mathbf{1}_{[t,\infty)}$ (non-decreasing). Then $\int u \, \mathrm{d}F = 1 - F(t) \geq 1 - G(t) = \int u \, \mathrm{d}G$, giving $F(t) \leq G(t)$.

(⇐): Integrate by parts: $\int u \, \mathrm{d}F - \int u \, \mathrm{d}G = \int [G(t) - F(t)] u'(\mathrm{d}t) \geq 0$ since $G(t) \geq F(t)$ and $u' \geq 0$. ■

Proposition: FOSD and Means

$F \succsim_1 G \Rightarrow \mu_F \geq \mu_G$. The converse fails: equal means do not imply FOSD.

If F FOSD G and $\mu_F = \mu_G$, then $F = G$ (equal means plus FOSD collapses to equality of distributions).

8.2 Second-Order Stochastic Dominance

Definition: SOSD and Mean-Preserving Spread

F *second-order stochastically dominates* G ($F \succsim_2 G$) if for every non-decreasing, weakly concave u:

$$\int u \, \mathrm{d}F \;\geq\; \int u \, \mathrm{d}G.$$

FOSD ⇒ SOSD, but not conversely. SOSD is the appropriate ranking for risk-averse agents.

G is a *mean-preserving spread (MPS)* of F if $G \overset{d}{=} X + Z$ with $X \sim F$ and $\mathbb{E}[Z \mid X] = 0$. Equivalently, $\int_{-\infty}^t G(s) \, \mathrm{d}s \geq \int_{-\infty}^t F(s) \, \mathrm{d}s$ for all t, with equality at $+\infty$.

Theorem: SOSD and Mean-Preserving Spreads

Suppose F and G have the same mean. Then $F \succsim_2 G \iff G$ is a mean-preserving spread of F.

8.3 Value of Information

Definition: Value of Information

Without signal: $V(\rho) := \sup_{a \in A} \mathbb{E}_\theta[u(a, \theta)]$.

With signal structure $\mathcal{I} = \{\ell(\cdot \mid \theta)\}_\theta$: choose strategy $\alpha : \mathcal{T} \to A$. Ex ante value:

$$EV(\rho, \mathcal{I}) := \mathbb{E}_t \left[\sup_{a \in A} \mathbb{E}_\theta[u(a, \theta) \mid t] \right].$$

Value of information: $\mathrm{VoI}(\rho, \mathcal{I}) := EV(\rho, \mathcal{I}) - V(\rho) \geq 0$.

Non-negativity proof: $EV(\rho, \mathcal{I}) = \mathbb{E}_t[\sup_a \mathbb{E}_\theta[u \mid t]] \geq \sup_a \mathbb{E}_t[\mathbb{E}_\theta[u \mid t]] = V(\rho)$, by Jensen (since sup is convex). ■

8.4 Garbling and the Blackwell Ordering

Definition: Garbling

$\mathcal{I}'$ is a *garbling* of $\mathcal{I}$ if $\ell'(s' \mid \theta) = \int G(s' \mid s)\, \ell(\mathrm{d}s \mid \theta)$ for some state-independent kernel G. Garbling makes the signal noisier by applying additional, state-independent randomization.

Theorem: Blackwell Ordering — TFAE

"$\mathcal{I}$ is weakly more informative than $\mathcal{I}'$" iff any of the following:

1. $\mathcal{I}'$ is a garbling of $\mathcal{I}$.
2. Every strategy implementable with $\mathcal{I}'$ is implementable with $\mathcal{I}$ (for every decision problem).
3. If F and F' are the distributions over posterior beliefs induced by $\mathcal{I}$ and $\mathcal{I}'$, then F is a mean-preserving spread of F'.
4. F dominates F' in convex order: $\int \varphi \, \mathrm{d}F \geq \int \varphi \, \mathrm{d}F'$ for all convex φ.

Key insight: more informative signals induce posteriors that are more spread out (a MPS of less informative posteriors). This connects Blackwell ordering to SOSD of posterior distributions.

Garbling and SOSD

Let $\mathcal{V}$ be the distribution over expected utilities induced by $\mathcal{I}$. Then:

- V is a MPS of $\mathbb{E}[V \mid \mathcal{I}']$: finer signal induces more spread in posteriors.
- More spread in posteriors $\Leftrightarrow$ more informative signal.
- VoI is always non-negative by the convexity of the value function $\rho \mapsto V(\rho)$ (Jensen's inequality applied globally).