Microeconomic Theory I — Study Note

Chapter 9: Social Choice and Welfare

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9 Social Choice and Welfare

Roadmap: Money metric utility → Social welfare functionals → May's Theorem (2 alternatives) → Arrow's Impossibility Theorem → Ranked choice voting → Median Voter Theorem → Harsanyi's Theorem.

9.1 Money Metric Utility

Definition: Money Metric Utility

Fix a reference price vector p_{ref}. The *money metric utility* of consumption bundle x at prices p is:

$$M(p, x) := e\big(p,\, v(p_{\text{ref}}, x)\big).$$

It measures, in monetary terms at prices p, the welfare of consuming x.
Well-behaved condition (Blackorby–Donaldson): $M(p, x)$ is well-behaved if $M(p, x) = a(p)+ b(p)\, v(p_{\text{ref}}, x)$ for $b > 0$. Otherwise, $M(p, x)$ may fail concavity in x for some p, undermining social welfare comparisons: an egalitarian redistribution need not improve the aggregate money metric even when each individual's metric is concave.

9.2 Social Welfare Functionals

Definition: Social Welfare Functional

Setting: alternatives $\mathcal{Y}$ with $|\mathcal{Y}| \geq 2$; n individuals each with preference $\succsim_i$ over $\mathcal{Y}$; a *social welfare functional* $F(\{\succsim_i\}) = \succsim_{\text{soc}}$.

9.3 Two Alternatives: May's Theorem

Theorem: May's Theorem

With $|\mathcal{Y}| = 2$, majority rule is the *unique* social choice rule satisfying:

1. **Anonymity:** for any permutation π of individuals, $F(\{\succsim_i\}) = F(\{\succsim_{\pi(i)}\})$.
2. **Neutrality:** the rule treats all alternatives symmetrically.
3. **Positive responsiveness:** if F ranks $x \succsim_{\text{soc}} y$ and some individual flips from $y \succ_i x$ to $x \succ_i y$, then $x \succ_{\text{soc}} y$ strictly.

Proof sketch

Anonymity: the outcome depends only on the *number* of agents preferring x to y (not which agents). Neutrality: the rule treats x and y symmetrically. Positive responsiveness pins down the threshold: the only anonymous, neutral, positively responsive rule is majority rule (simple majority). Any other threshold (e.g. supermajority) violates positive responsiveness or neutrality. ■

Condorcet Cycles with ≥ 3 Alternatives

With ≥ 3 alternatives, majority rule need not be rational. Three voters: $x \succ_1 y \succ_1 z$, $y \succ_2 z \succ_2 x$, $z \succ_3 x \succ_3 y$. Majority rule gives $x \succ y$, $y \succ z$, $z \succ x$ — a cycle, violating transitivity.

Condorcet winner: x^* beats every other alternative in pairwise majority voting. A Condorcet winner need not exist (as the cycle shows).

9.4 Arrow's Impossibility Theorem

Theorem: Arrow's Impossibility Theorem

Suppose $|\mathcal{Y}| \geq 3$ and F satisfies:

1. **Unrestricted domain:** F is defined for all preference profiles.
2. **Weak Pareto:** if every individual strictly prefers x to y, then $x \succ_{\text{soc}} y$.
3. **IIA:** if two profiles agree on the ranking of x vs. y for all individuals, then F agrees on x vs. y.

Then F is a **dictatorship**: $\exists j$ s.t. $x \succ_j y \Rightarrow x \succ_{\text{soc}} y$ for all x, y.

Proof sketch

IIA and Pareto together imply the existence of a *pivotal voter*: an individual j whose preference reversal from $y \succ_j x$ to $x \succ_j y$ flips the social ordering. One then shows this pivotal voter is decisive over *every* pair of alternatives (applying IIA and Pareto repeatedly), making them a dictator. The key step: with ≥ 3 alternatives, IIA forces transitivity to propagate decisiveness across all pairs. ■

Ranked Choice Voting and IIA Violation

Three candidates D, M, R: Group 1 (35%): $D \succ M \succ R$; Group 2 (20%): $M \succ R \succ D$; Group 3 (48%): $R \succ D \succ M$. No majority first. Eliminate M (20%); Group 2 votes go to R. Final: R wins with 68%.

Violation of IIA: candidate A could strategically change their preference over M and D to prevent R from winning — the relative ranking of R vs. D depends on the presence of M, an "irrelevant" alternative.

9.5 Median Voter Theorem

Theorem: Median Voter Theorem

Suppose:

1. All alternatives are ordered on a line (single-dimensional policy space).
2. Each individual has a unique *ideal point* and *single-peaked* preferences: alternatives are weakly

worse as they move away from the ideal point.

Then the *median voter's* ideal point is a Condorcet winner.

Borda Count and Modified IIA

Under *modified IIA* (IIA only needs to hold among profiles sharing the same set of alternatives), the **Borda count** is the unique social choice rule satisfying Anonymity, Neutrality, and modified IIA (Maskin). Borda count: assign rank scores, sum across agents.

9.6 Cardinal Social Welfare: Harsanyi's Theorem

Theorem: Harsanyi's Theorem

Assume finitely many alternatives X, n individuals with vNM utility $U_i : X \to \mathbb{R}$, and a social welfare function $W : X \to \mathbb{R}$ satisfying the vNM axioms.
Under **semistrong Pareto** ($W(x) \geq W(y) \Leftrightarrow \forall\, i : U_i(x) \geq U_i(y)$):

$$W(x) = a + \sum_i \lambda_i U_i(x) \quad \text{for some } \lambda_i \geq 0.$$

Under **strong Pareto** (strict for at least one j): all $\lambda_i > 0$.
Social welfare must be a positive affine combination of individual vNM utilities.

Proof sketch

W and $\{U_i\}$ all satisfy vNM axioms $\Rightarrow$ all are linear in probabilities. Semistrong Pareto: W is zero on the null space of $\sum_i U_i$. By a separating hyperplane argument in the space of utility vectors $(U_1(x), \ldots, U_n(x))$, W must lie in the cone spanned by $\{U_i\}$, giving the weighted sum. Strong Pareto forces all weights strictly positive. ■

Harsanyi vs. Arrow

Harsanyi: requires *cardinal* (vNM) individual utilities and shows the social welfare function must be a weighted utilitarian sum. The weights λ_i are not determined by the theorem — their choice reflects normative judgments about interpersonal comparisons.
Arrow: works with *ordinal* preferences only. The impossibility result says no aggregation rule can simultaneously satisfy Pareto, IIA, and non-dictatorship when $|\mathcal{Y}| \geq 3$.