Real Analysis II: Study Note

Chapter 1: Introduction

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1 Measure Spaces

Definition 1.1: Measure Space

A **measure space** is a triple $(X, \mathcal{F}, \mu)$ where:

- X is a set.
- $\mathcal{F}$ is a σ**-algebra** of subsets of X: contains X, and is closed under complements and countable unions (closure under countable intersections follows by De Morgan's laws and is hence redundant).
- $\mu : \mathcal{F} \to [0, \infty]$ is a **measure**: satisfying $\mu(\varnothing) = 0$ and countable additivity—for any pairwise disjoint $\{A_n\}_{n \in \mathbb{N}} \subset \mathcal{F}$,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Examples.

(i) **Counting Measure.** X any set, $\mathcal{F} = 2^X$, $\mu(A) = |A|$ (cardinality).

(ii) **Point (Dirac) Measure.** Fix $p \in X$:

$$\delta_p(A) = \begin{cases} 1 & p \in A, \\ 0 & p \notin A. \end{cases}$$

(iii) **Lebesgue Measure.** $X = \mathbb{R}^d$; the unique locally finite translation-invariant Borel measure normalized by $m([0,1]^d) = 1$.

2 Lebesgue Outer Measure

Definition 2.1: Rectangle in $\mathbb{R}^d$

A **rectangle** $R \subset \mathbb{R}^d$ is a product of bounded closed intervals:

$$R = I_1 \times \cdots \times I_d, \quad I_k = [a_k, b_k], \qquad |R| := \prod_{k=1}^{d} (b_k - a_k).$$

Definition 2.2: Lebesgue Outer Measure

For any $A \subset \mathbb{R}^d$,

$$m^*(A) := \inf \left\{ \sum_{j=1}^{\infty} |R_j| : A \subset \bigcup_{j=1}^{\infty} R_j,\ R_j \text{ closed rectangles} \right\}.$$

Definition 2.3: Almost Disjoint

A collection of rectangles $\{R_j\}$ is **almost disjoint** if their interiors are pairwise disjoint.

Lemma 2.4

$m^*(E) = 0$ for any countable set $E \subset \mathbb{R}^d$.

Lemma 2.5

If a rectangle $R = \bigcup_{j=1}^{N} R_j$ (finite, almost disjoint), then $|R| = \sum_{j=1}^{N} |R_j|$.

Lemma 2.6

If $\{R_j\}$ is almost disjoint with $R = \bigcup_j R_j$, then $|R| = \sum_j |R_j|$.

Proposition 2.7

For a closed rectangle $R \subset \mathbb{R}^d$, $m^*(R) = |R|$.

Remark 2.1. *The boundary of a rectangle has outer measure zero, so open and closed rectangles have the same outer measure.*

Proposition 2.8

For any $E \subset \mathbb{R}^d$ and $\varepsilon > 0$, there is a countable covering $\{Q_j\}$ of E by closed cubes with $\sum_j m^*(Q_j) \leq m^*(E) + \varepsilon$.

Further observations.

(i) Open set equivalence. $m^*(E) = \inf\{m^*(O) : O \supset E,\ O \text{ open}\}$.

(ii) Metric outer measure. On a metric space (X, d), an outer measure satisfying $d(A, B) > 0 \Rightarrow \mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$ is called a **metric outer measure**. Recall $d(A, B) = \inf_{a \in A, b \in B} d(a, b)$.

Lemma 2.9

m^* is a metric outer measure on $\mathbb{R}^d$.

Theorem 2.10: Open Set Decomposition[1]

Every open set $O \subset \mathbb{R}^d$ is a countable almost disjoint union of closed cubes:

$$O = \bigcup_{i=1}^{\infty} R_i \quad \text{and} \quad m^*(O) = \sum_{i=1}^{\infty} m^*(R_i).$$

2.1 The Cantor Set

The **Cantor set** is constructed iteratively:

$$\begin{aligned} C_0 &= [0,1], \\ C_1 &= [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1], \\ C_k &= C_{k-1} \setminus \{\text{open middle third of each interval in } C_{k-1}\}, \quad k \geq 1, \end{aligned}$$

and $\mathcal{C} := \bigcap_{k=0}^{\infty} C_k$.

Properties of $\mathcal{C}$.

- **Compact and non-empty**: each C_k is non-empty and compact, with $C_k \supset C_{k+1}$, so $\mathcal{C} = \bigcap_k C_k \neq \varnothing$ by the finite intersection property; $\mathcal{C}$ is closed and bounded.
- **Totally disconnected**: $\forall\, x \neq y \in \mathcal{C}$, $\exists\, z \in (x,y)$ with $z \notin \mathcal{C}$.
- **Perfect**: closed with no isolated points.

"Size" of $\mathcal{C}$. (1) $|\mathcal{C}|$ is a continuum (bijection $\mathcal{C} \to [0,1]$). (2) $m(\mathcal{C}) = 0$: since $\mathcal{C} \subset C_k$, $m(\mathcal{C}) \leq m(C_k) = (2/3)^k \to 0$.

3 General Outer Measures

Definition 3.1: Outer Measure

On any set X, an **outer measure** is $\mu^* : \mathcal{P}(X) \to [0,\infty]$ satisfying:

(i) $\mu^*(\varnothing) = 0$.

(ii) **Monotonicity:** $A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$.

(iii) **Countable sub-additivity:** $\mu^*\left(\bigcup_j A_j\right) \leq \sum_j \mu^*(A_j)$.

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[1] The standard proof partitions O into dyadic cubes: at each scale 2^{-k}, include a dyadic cube if it is contained in O and not already covered by a coarser cube. This yields an almost disjoint family. The measure identity then follows from Lemma 2.6.

Theorem 3.2

The Lebesgue outer measure m^* is an outer measure on $\mathbb{R}^d$.

4 Lebesgue Measurable Sets

Definition 4.1: Lebesgue Measurable Set[2]

$E \subset \mathbb{R}^d$ is **Lebesgue measurable** if for every $\varepsilon > 0$, there exists an open set $U \supset E$ with $m^*(U \setminus E) < \varepsilon$. The collection of all measurable sets is denoted $\mathcal{M}$.

Lemma 4.2: Equivalent Characterization

$E \subset \mathbb{R}^d$ is Lebesgue measurable if and only if for every $\varepsilon > 0$, there exists a closed set $F \subset E$ with $m^*(E \setminus F) < \varepsilon$.

Proof idea: take complements; E^c *is also measurable.*

Lemma 4.3

If $m^*(E) < \infty$, then for every $\varepsilon > 0$ there exists open $U \supset E$ with $m^*(U) \leq m^*(E) + \varepsilon$.

The definition implies this lemma, but not conversely. If m^* *were additive on all sets, measurability would be vacuous—but such additivity fails (cf. Vitali sets, Axiom of Choice).*

Definition 4.4: Carathéodory Condition[3]

$E \subset X$ is **Carathéodory measurable** w.r.t. μ^* if for all $A \subset X$,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

Theorem 4.5

$\mathcal{M}$ is a σ-algebra.

Definition 4.6: Lebesgue Measure

The **Lebesgue measure** is the restriction $m := m^*|_{\mathcal{M}}$.

The outer measure m^* *is defined on all subsets of* $\mathbb{R}^d$ *but fails to be countably additive on all subsets. Restricting to* $\mathcal{M}$ *recovers full additivity.*

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[2] E is measurable if it is approximable from outside by open sets. Open sets are measurable by definition. $\mathcal{M}$ will be shown to be a σ-algebra (Theorem 4.5). Given a topology, one recovers μ from $\mathcal{M}$.

[3] This defines a "good set" by how it *exactly* partitions every subset of X under μ^*. No topology is needed, making it strictly broader than Definition 4.1. Starting from μ^*, the condition automatically produces the σ-algebra $\mathcal{M}$.

Theorem 4.7

$m = m^*|_{\mathcal{M}}$ is a measure; in particular it is countably additive.

4.1 Continuity of Measure

Definition 4.8: Nested Sequences

For $\{E_k\} \subset \mathcal{M}$: $E_k \nearrow E$ means $E_i \subset E_j$ for $i < j$ and $E = \bigcup_k E_k$; $E_k \searrow E$ means $E_i \supset E_j$ for $i < j$ and $E = \bigcap_k E_k$.

Lemma 4.9: Continuity of Measure

For $\{E_k\} \subset \mathcal{M}$:

(i) $E_k \nearrow E \;\Rightarrow\; \lim_{k\to\infty} m(E_k) = m(E)$.

(ii) $E_k \searrow E$ and $m(E_1) < \infty \;\Rightarrow\; \lim_{k\to\infty} m(E_k) = m(E)$.

5 Borel Sets

For any $E \in \mathcal{M}$, given $\varepsilon > 0$:

(i) $\exists$ open $U \supset E$ with $m(U \setminus E) < \varepsilon$. Approximate by open sets from outside

(ii) $\exists$ closed $F \subset E$ with $m(E \setminus F) < \varepsilon$. Approximate by closed sets from inside

(iii) If $m(E) < \infty$: $\exists$ compact $K \subset E$ with $m(E \setminus K) < \varepsilon$.

Taking sequences: $A = \bigcap_n U_n$ (G_δ) and $B = \bigcup_n F_n$ (F_σ). Then $B \subset E \subset A$ with $m(A \setminus E) = m(E \setminus B) = 0$.

Remark 5.1. G_δ [4]*: countable intersection of open sets.* F_σ [5]*: countable union of closed sets.*

Definition 5.1: Borel σ-Algebra

The **Borel σ-algebra** $\mathcal{B}$ is the σ-algebra generated by all open sets. Its elements are **Borel sets**.

One has $\mathcal{B} \subset \mathcal{M}$, but $\mathcal{M} \not\subset \mathcal{B}$ (the Axiom of Choice produces measurable non-Borel sets).

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[4] G for *Gebiet* (German for "region"), δ for *Durchschnitt* (German for "intersection")

[5] F for *Fermé* (French for "closed"), σ for *Summe* (German for "sum")

Proposition 5.2

Given $E \in \mathcal{M}$, there exist $A, B \in \mathcal{B}$ with $B \subset E \subset A$ and $m(E \setminus B) = m(A \setminus E) = 0$.

Remark 5.2. $m^*(E) = 0 \Rightarrow E \in \mathcal{M}$. *Note that null sets need* not *be Borel: the Cantor set* $\mathcal{C}$ *has measure zero and contains subsets (obtainable via the Axiom of Choice) that are Lebesgue measurable but not Borel.*

Definition 5.3: Borel Measure

The **Borel measure** is Lebesgue measure restricted to $\mathcal{B}$.

6 Measurable Functions

Definition 6.1: Measurable Function[6]

$f : \mathbb{R}^d \to \mathbb{R}$ is **measurable** if for every $a \in \mathbb{R}$,

$$\{f > a\} := \{x \in \mathbb{R}^d : f(x) > a\} \in \mathcal{M}.$$

Level sets. Since $\mathcal{M}$ is a σ-algebra, $\{f > a\} \in \mathcal{M}$ propagates to all standard level sets via countable operations: $\{f \geq a\}, \{f < a\}, \{f = a\}, \{f \leq a\} \in \mathcal{M}$.

Proposition 6.2: TFAE for Measurability

For $f : \mathbb{R}^d \to \mathbb{R}$, the following are equivalent:

(i) f is measurable.

(ii) $f^{-1}(\text{open set}) \in \mathcal{M}$ for every open set.

(iii) $f^{-1}(\text{closed set}) \in \mathcal{M}$ for every closed set.

(iv) $f^{-1}(B) \in \mathcal{M}$ for every $B \in \mathcal{B}$.

Remark 6.1. *Condition (iv) does* not *extend to all of* $\mathcal{M}$*:* $f^{-1}(E) \notin \mathcal{M}$ *in general for* $E \in \mathcal{M}$*.*

Definition 6.3: Borel Function

f is a **Borel function** if $f^{-1}(\mathcal{B}) \subset \mathcal{B}$.

Proposition 6.4: Algebraic Stability

If $f, g : \mathbb{R}^d \to \mathbb{R}$ are measurable, so are af (any $a \in \mathbb{R}$), $f + g$, and fg.

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[6] Lebesgue integration approximates f via *level sets* (horizontal slices of the graph) rather than the domain partitions used in Riemann integration. This makes the theory robust under pointwise limits.

Proposition 6.5: Stability under Limits

If $\{f_k\}$ are measurable, so are $\sup_k f_k$, $\inf_k f_k$, $\limsup_k f_k$, $\liminf_k f_k$, and $\lim_k f_k$ (wherever it exists).

Proposition 6.6: Stability under Composition

If $f : \mathbb{R}^d \to \mathbb{R}$ is measurable and $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous (or Borel), then $\varphi \circ f$ is measurable.

Definition 6.7: Almost Everywhere

A property holds **almost everywhere** (a.e.) if the set of exceptions has measure zero (in probability: **almost surely**, a.s.). For functions: $f = g$ a.e. means $m(\{x : f(x) \neq g(x)\}) = 0$.

Theorem 6.8: Egorov's Theorem[7]

Let $E \in \mathcal{M}$ with $m(E) < \infty$ and $\{f_k\}$ measurable on E. If $f_k \to f$ a.e. on E, then for every $\varepsilon > 0$ there exists $A_\varepsilon \in \mathcal{M}$ with $A_\varepsilon \subset E$, $m(E \setminus A_\varepsilon) < \varepsilon$, and $f_k \to f$ *uniformly* on A_ε.

7 Littlewood's Three Principles

Remark 7.1 (Informal Statement)**.**

(i) Every measurable set is "nearly"[8] *a finite union of rectangles.*

(ii) Every measurable function is "nearly" continuous.

(iii) Every a.e.-convergent sequence is "nearly" uniformly convergent.

"Nearly" has no formal meaning; each statement is made precise below.

Theorem 7.1: Littlewood's First Principle

For $E \in \mathcal{M}$ with $m(E) < \infty$ and $\varepsilon > 0$, there exists a finite collection of closed rectangles $\{R_i\}_{i=1}^N$ such that

$$m\left(E \,\Delta \bigcup_{i=1}^{N} R_i\right) < \varepsilon,$$

where $A \,\Delta\, B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference.

Borel–Cantelli Lemma. For $\{E_k\} \subset \mathcal{M}$ with $\sum_k m(E_k) < \infty$: $m(\limsup_k E_k) = 0$.
Recall: $\chi_A = \mathbf{1}_A$*, and* $\limsup_k \chi_{E_k} = \chi_{\limsup_k E_k}$.

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[7] Egorov's theorem is Littlewood's Third Principle (Section 7). Shrink the domain slightly to convert a.e. convergence into uniform convergence. Integration extends well to a.e. limits, but differentiation does *not*—a key distinction.

[8] Note that "nearly" has no formal mathematical meanings, unlike "almost"

7.1 Approximation by Simple and Step Functions

Definition 7.2: Simple Function

A **simple function** is $\varphi(x) = \sum_{j=1}^{n} c_j \, \chi_{E_j}(x)$ with $E_j \in \mathcal{M}$, $m(E_j) < \infty$, $c_j \in \mathbb{R}$.

Definition 7.3: Step Function

A **step function** is $\psi(x) = \sum_{j=1}^{n} c_j \, \chi_{R_j}(x)$ with R_j rectangles.

Theorem 7.4: Simple Function Approximation

For any measurable $f : \mathbb{R}^d \to \mathbb{R}$, there exist simple functions $\{\varphi_n\}$ with $\varphi_n \to f$ pointwise. If $f \geq 0$, one may take $0 \leq \varphi_n \nearrow f$.

Theorem 7.5: Step Function Approximation

For any measurable $f : \mathbb{R}^d \to \mathbb{R}$, there exist step functions $\{\psi_n\}$ with $\psi_n \to f$ a.e.

Step functions are a coarser approximation than simple functions (rectangles vs. arbitrary measurable sets).

Theorem 7.6: Lusin's Theorem (Littlewood's Second Principle)

If f is measurable on $E \in \mathcal{M}$ with $m(E) < \infty$, then for every $\varepsilon > 0$ there exists closed $F \subset E$ with $m(E \setminus F) < \varepsilon$ such that $f|_F$ is continuous.

Remark 7.2. *Lusin's theorem does* not *say f is continuous on F as a subset of E. The set F is not the set of continuity points of f on $\mathbb{R}^d$.*
Counterexample: $\chi_{\mathbb{Q}} : \mathbb{R} \to \{0, 1\}$. *Taking $F = \mathbb{Q}^c$: $f|_F = 0$ is continuous on F, yet $\chi_{\mathbb{Q}}$ is nowhere continuous on $\mathbb{R}$.*

Proofs of Key Results

Proposition 2.7: $m^*(R) = |R|$

Proof. $(\leq)$ R covers itself, so $m^*(R) \leq |R|$.

$(\geq)$ Let $\{R_j\}$ be any countable covering of R. Fix $\varepsilon > 0$; enlarge each R_j to an open rectangle $S_j \supset R_j$ with $|S_j| = (1+\varepsilon)|R_j|$. By compactness of R, finitely many S_j cover R; Lemma 2.5 then gives

$$|R| \leq (1+\varepsilon)\sum_{j=1}^{N} |R_j| \leq (1+\varepsilon)\sum_{j=1}^{\infty} |R_j|.$$

Since $\varepsilon > 0$ is arbitrary, $|R| \leq \inf \sum |R_j| = m^*(R)$. ☐

Theorem 3.2: m^* is an outer measure

Proof. Empty-set and monotonicity are immediate. For *countable sub-additivity*: fix $A \subset \mathbb{R}^d$ and covering $\{A_j\}$. For each j choose rectangles $\{R_{jk}\}_k$ covering A_j with $\sum_k |R_{jk}| \leq m^*(A_j) + \varepsilon/2^j$. Then $\{R_{jk}\}_{j,k}$ covers A, giving $m^*(A) \leq \sum_{j,k} |R_{jk}| \leq \sum_j m^*(A_j) + \varepsilon$. ☐

Theorem 4.5: $\mathcal{M}$ is a σ-algebra

Proof. We establish five claims. Recall: E is measurable if $\forall \varepsilon > 0$, $\exists$ open $U \supset E$ with $m^*(U \setminus E) < \varepsilon$.

Claim 1 ($m^*(E) = 0 \Rightarrow E \in \mathcal{M}$): Take open $U \supset E$ with $m^*(U) < \varepsilon$; monotonicity gives $m^*(U \setminus E) < \varepsilon$.

Claim 2 (Countable unions): For $\{A_j\} \subset \mathcal{M}$, choose open $U_j \supset A_j$ with $m^*(U_j \setminus A_j) < \varepsilon/2^j$. Since $(U \setminus \bigcup A_j) \subset \bigcup (U_j \setminus A_j)$, sub-additivity gives $m^*(U \setminus \bigcup A_j) < \varepsilon$.

Claim 3 (Compact sets, hence all closed sets): For compact K, write $U \setminus K = \bigcup_i Q_i$. For $A_N = \bigcup_{i=1}^N Q_i$: $d(A_N, K) > 0$ (disjoint compact), so $m^*(A_N) + m^*(K) = m^*(A_N \cup K) \leq m^*(U) < m^*(K) + \varepsilon$, yielding $m^*(U \setminus K) < \varepsilon$.

Claim 4 (Complements): For $E \in \mathcal{M}$, choose $U_n \supset E$ with $m^*(U_n \setminus E) < 1/n$. Set $S = \bigcup_n U_n^c \in \mathcal{M}$ (Claims 2,3). Then $E^c \setminus S \subset U_n \setminus E$ so $m^*(E^c \setminus S) \leq 1/n \to 0$; by Claim 1, $E^c = S \cup (E^c \setminus S) \in \mathcal{M}$.

Claim 5 (Countable intersections): By Claims 2, 4, and De Morgan. ☐

Theorem 4.7: Countable additivity of m

Proof. Sub-additivity gives $m(E) \leq \sum_j m(E_j)$; it remains to show $\geq$.

Compact approximation lemma: If $m(E) < \infty$, then $\exists$ compact $F \subset E$ with $m(E \setminus F) < \varepsilon$. (Take closed $F' \subset E$ with $m(E \setminus F') < \varepsilon/2$, intersect with ball $B(0, r)$; by continuity of measure

$m(F' \setminus B(0,r)) \to 0$, so choose r large enough.)

Given $E = \bigsqcup_j E_j$, choose compact $K_j \subset E_j$ with $m(E_j \setminus K_j) \leq \varepsilon/2^j$. The K_j are pairwise disjoint compact sets, so $d(K_i, K_j) > 0$ and by metric outer measure additivity:

$$m(E) \geq \sum_{j=1}^{N} m(K_j) \geq \sum_{j=1}^{N} \big[m(E_j) - \tfrac{\varepsilon}{2^j}\big] \quad \forall N.$$

Let $N \to \infty$ and $\varepsilon \to 0$. ☐

Lemma 4.9: Continuity of measure

Proof. **(i) Upward.** Decompose $E = E_1 \cup (E_2 \setminus E_1) \cup \cdots$ (disjoint). By countable additivity: $m(E) = m(E_1) + \sum_{k=1}^{\infty}[m(E_{k+1}) - m(E_k)] = \lim_k m(E_k)$.

(ii) Downward. Write $E_k = E \cup (E_k \setminus E)$ with $E_k \setminus E = \bigsqcup_{j=k}^{\infty}(E_j \setminus E_{j+1})$. Then $m(E_k) = m(E) + \sum_{j=k}^{\infty} m(E_j \setminus E_{j+1})$. Since the full series $\leq m(E_1) < \infty$, the tail $\to 0$. ☐

Theorem 7.1: Littlewood's First Principle

Proof. Choose rectangles $\{R_i\}$ covering E with $\sum m(R_i) \leq m(E) + \varepsilon/2$. Since $m(E) < \infty$, choose N with $\sum_{i>N} m(R_i) < \varepsilon/2$. Let $R_N = \bigcup_{i=1}^{N} R_i$. Then:

$$m(E \,\Delta\, R_N) \leq m(R_N \setminus E) + m(E \setminus R_N) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

☐

Theorem 6.8: Egorov's Theorem

Proof. For $n, k \in \mathbb{N}$ set $E_{n,k} = \bigcup_{j \geq k}\{|f_j - f| \geq 1/n\}$. Since $f_k \to f$ a.e., $m(\bigcap_k E_{n,k}) = 0$. Note $E_{n,1} \subset E$ so $m(E_{n,1}) \leq m(E) < \infty$; since $E_{n,k} \searrow \bigcap_k E_{n,k}$, by Lemma 4.9(ii), $m(E_{n,k}) \to 0$; choose k_n with $m(E_{n,k_n}) < \varepsilon/2^n$. Set $A_\varepsilon = E \setminus \bigcup_n E_{n,k_n}$; then $m(E \setminus A_\varepsilon) < \varepsilon$, and on A_ε all $|f_j - f| < 1/n$ for $j \geq k_n$, i.e. $f_j \to f$ uniformly. ☐

Theorem 7.6: Lusin's Theorem

Proof. Approximate f by simple functions $\varphi_n \to f$ pointwise (Theorem 7.4). By Egorov, find $A \subset E$ with $m(E \setminus A) < \varepsilon/2$ on which $\varphi_n \to f$ uniformly. Each φ_n is a finite linear combination of indicator functions of measurable sets; by inner regularity, each such set is approximable from inside by a closed set, so φ_n is continuous on a closed subset of A of measure $> m(A) - \varepsilon/2^{n+1}$. Taking a further closed subset $F \subset A$ with $m(E \setminus F) < \varepsilon$, on F the functions $\{\varphi_n|_F\}$ are continuous and converge uniformly to $f|_F$, so $f|_F$ is continuous. ☐

Remark 7.3. *The proof sketch above omits details on controlling continuity of each* $\varphi_n|_F$*. A complete argument additionally invokes Tietze's extension theorem.*