
Real Analysis II: Study Note

Chapter 2: Integration Theory

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1 Integration of Simple Functions

The canonical failure of Riemann integration is $\chi_{\mathbb{Q}} : [0, 1] \rightarrow \mathbb{R}$: dense irrationals force every upper sum to 1 and every lower sum to 0, so the two never meet.

Roadmap: BCT \rightarrow Fatou \rightarrow MCT \rightarrow DCT

The central question throughout is: *when can we pass a limit through an integral?* Each theorem relaxes the price of admission. BCT imposes the most rigid constraints — a uniform bound M on the functions and a finite-measure domain, essentially trapping everything in a box. Fatou's Lemma then allows the functions to be unbounded, at the cost of only an inequality. MCT trades the uniform bound for monotonicity, letting the functions grow so long as they do so in an orderly fashion. DCT is the culmination: we need only a single integrable dominating function, making it the workhorse of the subject.

Definition 1.1: Integral of a Simple Function

Given a simple function $\varphi = \sum_{i=1}^k c_i \chi_{E_i}$, define

$$\int \varphi := \sum_i c_i m(E_i).$$

More generally, against a measure μ : $\int \varphi d\mu$; with explicit variable: $\int \varphi(x) d\mu(x)$; over a sub-domain E :

$$\int_E \varphi := \sum_i c_i m(E_i \cap E).$$

Proposition 1.2: Properties of the Simple Integral¹

For simple functions:

- (i) **Independence of representation.** If $\varphi = \sum_i c_i \chi_{E_i} = \sum_j b_j \chi_{F_j}$, then $\sum_i c_i m(E_i) = \sum_j b_j m(F_j)$.
- (ii) **Monotonicity.** $\varphi_1 \leq \varphi_2 \Rightarrow \int \varphi_1 \leq \int \varphi_2$.
- (iii) **Linearity.** $\int (a_1 \varphi_1 + a_2 \varphi_2) = a_1 \int \varphi_1 + a_2 \int \varphi_2$.
- (iv) **Triangle inequality.** $|\int \varphi| \leq \int |\varphi|$.

¹Property (i) is non-trivial and must be verified: the integral must not depend on the particular way φ is written as a linear combination of indicators.

2 Integration of Bounded Functions

Strategy: (i) approximate f by simple functions; (ii) use Egorov's theorem to ensure convergence of integrals.

Lemma 2.1

Let f be bounded on E with $m(E) < \infty$, and let $\{\varphi_j\}_{j \in \mathbb{N}}$ be simple functions with $\varphi_j \rightarrow f$ on E . Then:

- (i) If $|\varphi_j| \leq |f|$, then $\lim_{j \rightarrow \infty} \int_E \varphi_j$ exists.
- (ii) If $f = 0$ a.e., then $\lim_{j \rightarrow \infty} \int_E \varphi_j = 0$.

Definition 2.2: Support

The **support** of f is $\text{supp}(f) := \{f \neq 0\}$.

Definition 2.3: Lebesgue Integral — Bounded Case²

Given $f : \mathbb{R}^d \rightarrow \mathbb{R}$ bounded with $m(\text{supp}(f)) < \infty$, define

$$\int f := \lim_{n \rightarrow \infty} \int \varphi_n$$

for any sequence of simple functions $\{\varphi_n\}$ satisfying: $\varphi_n \rightarrow f$, $\text{supp}(\varphi_n) \subset \text{supp}(f)$, $|\varphi_n| \leq |f|$.

Example. $\chi_{\mathbb{Q}} : [0, 1] \rightarrow \{0, 1\}$ is *not* Riemann integrable (dense discontinuities everywhere). However, $\int_{[0,1]} \chi_{\mathbb{Q}} = 0$ in the Lebesgue sense, since \mathbb{Q} has measure zero.

Theorem 2.4: Bounded Convergence Theorem³

Suppose $\{f_n\}_{n \in \mathbb{N}}$ satisfies $|f_n| \leq M$ and $\text{supp}(f_n) \subset E$ for all n , with $m(E) < \infty$. If $f_n \rightarrow f$ a.e. on E , then

$$\lim_{n \rightarrow \infty} \int_E |f_n - f| = 0, \quad \text{and in particular} \quad \lim_{n \rightarrow \infty} \int f_n = \int f.$$

²Two facts must be checked: (i) the limit exists (by Lemma 2.1); (ii) $\int f$ is independent of the choice of approximating sequence $\varphi_n \rightarrow f$.

³Prototype for Lebesgue's DCT (Theorem 5.3). The BCT constrains f_n in a rectangular box $M \cdot \chi_E$. The DCT replaces this rigid box with any integrable "envelope" g .

3 Integration of Positive Functions

Definition 3.1: Lebesgue Integral — Positive Case

For $f : \mathbb{R}^d \rightarrow [0, \infty]$ (i.e., $f \geq 0$), define

$$\int f := \sup \left\{ \int g : g \text{ bounded, } m(\text{supp}(g)) < \infty, 0 \leq g \leq f \right\}.$$

Proposition 3.2

The integral $\int f$ is monotone and linear over positive functions.

4 The Full Lebesgue Integral

Definition 4.1: Lebesgue Integral — General Case

Decompose any $f = f^+ - f^-$, where $f^+ := \max(f, 0)$ and $f^- := \max(-f, 0)$. Define

$$\int f := \int f^+ - \int f^-,$$

provided both $\int f^+$ and $\int f^-$ are finite.

$f : E \rightarrow \mathbb{R}$ is **integrable** if $\int |f| < \infty$, equivalently $\int f^+ < \infty$ and $\int f^- < \infty$.

Extension to \mathbb{C} . For $f : \mathbb{R}^d \rightarrow \mathbb{C}$, write $f = u + iv$ where $u = \text{Re}(f)$ and $v = \text{Im}(f)$ are real-valued. Define $\int f := \int u \, dm + i \int v \, dm$, integrating u and v via the real decomposition $u = u^+ - u^-$, $v = v^+ - v^-$. Say f is integrable if $\int |f| < \infty$ (equivalently, both u and v are integrable).

Proposition 4.2

If f is (real-valued) integrable, then $\int f$ is linear and monotone. For complex-valued integrable f , $\int f$ is linear and satisfies $|\int f| \leq \int |f|$.

Remark 4.1. *Nothing in Sections 1–4 is specific to Lebesgue measure; the construction works for any measure μ on any measure space (X, \mathcal{F}, μ) .*

Theorem 4.3: Markov's Inequality (Chebyshev's Inequality)

For $f \geq 0$ and $\alpha > 0$,

$$m(\{f \geq \alpha\}) \leq \frac{1}{\alpha} \int f.$$

This is sometimes called **Markov's inequality**. The name **Chebyshev's inequality** is more commonly reserved for the p -th power extension:

$$m(\{f \geq \alpha\}) = m(\{f^p \geq \alpha^p\}) \leq \frac{1}{\alpha^p} \int f^p,$$

which in probability reads $P(|X - \mu| \geq k\sigma) \leq 1/k^2$ for $p = 2$.^a

^aIn probability: $P(X \geq a) \leq \mathbb{E}[X]/a$ for $a > 0$ is Markov's inequality; the variance form $P(|X - \mu| \geq k\sigma) \leq 1/k^2$ is Chebyshev's.

Two immediate corollaries:

Lemma 4.4

For $f : \mathbb{R}^d \rightarrow [0, \infty]$, if $\int f < \infty$, then $f < \infty$ a.e.

Lemma 4.5

For $f : \mathbb{R}^d \rightarrow [0, \infty]$, if $\int f = 0$, then $f = 0$ a.e.

5 Convergence Theorems

Central question: When does $f_n \rightarrow f$ imply $\int f_n \rightarrow \int f$, i.e. when can we exchange lim and \int ?

Theorem 5.1: Fatou's Lemma

If $f_n \geq 0$ for all n , then

$$\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n.$$

In particular, if $f_n \rightarrow f$ a.e. and $f_n \geq 0$, then $\int f \leq \liminf_{n \rightarrow \infty} \int f_n$.

Remark 5.1. The inequality can be strict. See Counterexample (iii) below for a sequence where $\int f_n = 1$ for all n yet $\int f = 0$.

Theorem 5.2: Monotone Convergence Theorem (MCT)

Let $f_k : \mathbb{R}^d \rightarrow [0, \infty]$ with $f_k \geq 0$. If $f_k \nearrow f$ monotonically (pointwise a.e.), then

$$\int f_k \longrightarrow \int f.$$

Corollary to MCT. If $u_n \geq 0$ for all n , then

$$\sum_{n=1}^{\infty} \int u_n = \int \sum_{n=1}^{\infty} u_n,$$

since the partial sums $\sum_{k=1}^n u_k$ increase monotonically.

Weighted Lebesgue measure. Given $f \geq 0$, define a new set function by

$$\mu(E) := \int_E f \, dm.$$

By MCT, μ is countably additive, hence a measure. It is called a **weighted Lebesgue measure** and written $\mu = f \, dm$.⁴

Theorem 5.3: Lebesgue Dominated Convergence Theorem (DCT)

If $f_n \rightarrow f$ a.e. and $|f_n| \leq g$ for some integrable g , then

$$\lim_{n \rightarrow \infty} \int |f_n - f| = 0, \quad \text{and in particular} \quad \int f_n \rightarrow \int f.$$

5.1 Common Counterexamples

Try to avoid the following pathologies when applying convergence theorems.

- (i) **Escape to infinity.** $f_n = \frac{1}{n} \chi_{[0,n]} : \mathbb{R} \rightarrow \mathbb{R}$. Then $f_n \rightarrow 0$ uniformly and $\int f_n = 1$ for all n , but $\int \lim f_n = 0$. (*Mass escapes to $+\infty$.*)
- (ii) **Concentrating spike.** $f_n = n \cdot \chi_{[0,1/n]}$. Then $\int f_n = 1$ for all n , but $f_n \rightarrow 0$ a.e. (*Mass concentrates at the origin.*)
- (iii) **Sliding block (Fatou strict).** $f_n = \chi_{[n,n+1]}$. Then $\int f_n = 1$ for all n , $f_n \rightarrow 0$ pointwise everywhere, yet $\int \lim f_n = 0 < 1 = \liminf \int f_n$. (*This shows Fatou's inequality can be strict: mass escapes to $+\infty$.*)

6 Convergence in Measure

Definition 6.1: Convergence in Measure

$f_n \rightarrow f$ **in measure** if for all $\varepsilon > 0$,

$$m(\{|f_n - f| > \varepsilon\}) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Relation to a.e. convergence.

- $f_n \xrightarrow{\text{a.e.}} f \Rightarrow f_n \xrightarrow{m} f$ (when $m(E) < \infty$, by Egorov's theorem).

⁴Prelude to how every continuous probability measure is defined.

- The converse fails: $f_n \xrightarrow{m} f \not\Rightarrow f_n \xrightarrow{\text{a.e.}} f$. (Exercise: the sliding block above converges in measure but not a.e.)

Remark 6.1. A sequence can converge a.e. to 0 while failing to converge in measure: consider the typewriter sequence on $[0, 1]$ — indicators of shrinking, cycling subintervals — which converges in measure to 0 but at no point a.e. Conversely, Counterexample (i) (escape to infinity) shows that a.e. convergence does not imply convergence in measure on infinite-measure spaces: $f_n = \frac{1}{n}\chi_{[0,n]} \rightarrow 0$ a.e., but $m(\{f_n > \varepsilon\}) = n(1 - n\varepsilon)^+ \rightarrow \infty$ for small ε . By Egorov's theorem, if $f_n \rightarrow f$ a.e. on E with $m(E) < \infty$, one can always find a large subset $A_\varepsilon \subset E$ on which $f_n \rightarrow f$ uniformly, giving $f_n \rightarrow f$ in measure on E .

7 Properties of Integrability

Proposition 7.1: Concentration of Integrability⁵

If f is integrable on \mathbb{R}^d , then for every $\varepsilon > 0$ there exists $E \subset \mathbb{R}^d$ with $m(E) < \infty$ such that

$$\int_{E^c} |f| < \varepsilon.$$

Proposition 7.2: Absolute Continuity of the Integral⁶

If f is integrable on \mathbb{R}^d , then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $E \in \mathcal{M}$ with $m(E) < \delta$,

$$\int_E |f| < \varepsilon.$$

Example (MCT application). For $\alpha > 1$, using the Gamma function identity $\frac{1}{n^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-nt} dt$ and MCT (all terms non-negative, partial sums increase):

$$\zeta(\alpha) = \sum_{n=1}^{\infty} \frac{1}{n^\alpha} = \frac{1}{\Gamma(\alpha)} \sum_{n=1}^{\infty} \int_0^\infty t^{\alpha-1} e^{-nt} dt \stackrel{\text{MCT}}{=} \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \frac{e^{-t}}{1 - e^{-t}} dt = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{t^{\alpha-1}}{e^t - 1} dt.$$

(MCT justifies interchanging the sum and integral since all terms are non-negative and partial sums increase monotonically to the full series.)

⁵Intuitively: integrable functions cannot spread their mass across sets of infinite measure. Proof via MCT: the truncated functions $|f|\chi_{\{|x|<N\}}$ increase monotonically to $|f|$, so their integrals converge to $\int |f|$ by MCT.

⁶Proof via contradiction + Borel–Cantelli + DCT. Assuming no such δ exists produces sets E_n with $m(E_n) < 1/2^n$ and $\int_{E_n} |f| \geq \varepsilon$. By Borel–Cantelli, $m(\limsup E_n) = 0$, so $\int_{\limsup E_n} |f| = 0$. But DCT gives $\int_{\cup_{k \geq n} E_k} |f| \rightarrow 0$, contradicting $\int_{E_n} |f| \geq \varepsilon$.

Proofs of Key Results

Theorem 4.3: Chebyshev's Inequality

Proof. By monotonicity of the integral and the fact that $f \geq \alpha$ on $\{f \geq \alpha\}$:

$$\int f \geq \int_{\{f \geq \alpha\}} f \geq \alpha \cdot m(\{f \geq \alpha\}).$$

Dividing by $\alpha > 0$ gives the result. □

Lemma 4.4: $\int f < \infty \Rightarrow f < \infty$ a.e.

Proof. By Chebyshev (Markov), for each $n \in \mathbb{N}$: $m(\{f \geq n\}) \leq \frac{1}{n} \int f \rightarrow 0$. The sets $\{f \geq n\}$ decrease to $\{f = +\infty\}$, but they may have infinite measure, so we cannot directly apply downward continuity. Instead, note that for any $R > 0$, $m(\{f = +\infty\} \cap B(0, R)) \leq m(\{f \geq n\} \cap B(0, R)) \leq \frac{1}{n} \int f \rightarrow 0$. Since R is arbitrary and $m(\{f = +\infty\}) = \sup_R m(\{f = +\infty\} \cap B(0, R))$, we conclude $m(\{f = +\infty\}) = 0$. □

Lemma 4.5: $\int f = 0 \Rightarrow f = 0$ a.e.

Proof. For each $n \in \mathbb{N}$, Chebyshev gives $m(\{f \geq 1/n\}) \leq n \int f = 0$. The sets $\{f \geq 1/n\} \nearrow \{f > 0\}$; by upward continuity of measure, $m(\{f > 0\}) = \lim_{n \rightarrow \infty} m(\{f \geq 1/n\}) = 0$. □

Theorem 2.4: Bounded Convergence Theorem

Proof. Fix $\varepsilon > 0$. By Egorov's theorem (applicable since $m(E) < \infty$), there exists $A_\varepsilon \subset E$ with $m(E \setminus A_\varepsilon) < \varepsilon$ on which $f_n \rightarrow f$ uniformly.

Split the integral:

$$\int_E |f_n - f| = \int_{A_\varepsilon} |f_n - f| + \int_{E \setminus A_\varepsilon} |f_n - f| \leq m(E) \cdot \sup_{A_\varepsilon} |f_n - f| + 2M \cdot m(E \setminus A_\varepsilon).$$

The first term $\rightarrow 0$ as $n \rightarrow \infty$ by uniform convergence on A_ε ; the second term is $\leq 2M\varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\lim_{n \rightarrow \infty} \int_E |f_n - f| = 0$. The convergence $\int f_n \rightarrow \int f$ then follows by the triangle inequality $|\int f_n - \int f| \leq \int |f_n - f|$. □

Theorem 5.1: Fatou's Lemma

Proof. Let $g_n := \inf_{k \geq n} f_k$, so $g_n \geq 0$, $g_n \nearrow \liminf_n f_n$ pointwise. Let g be any bounded function with $m(\text{supp}(g)) < \infty$ and $0 \leq g \leq \liminf_n f_n$. Set $h_n := \min\{g, g_n\}$; then $h_n \nearrow g$ pointwise, $|h_n| \leq \sup g =: M < \infty$, and $\text{supp}(h_n) \subset \text{supp}(g)$.

By the Bounded Convergence Theorem, $\int h_n \rightarrow \int g$. Since $h_n \leq g_n \leq f_k$ for all $k \geq n$, we have $\int h_n \leq \int f_k$ for all $k \geq n$, hence $\int h_n \leq \inf_{k \geq n} \int f_k$. Taking $n \rightarrow \infty$:

$$\int g = \lim_n \int h_n \leq \liminf_{n \rightarrow \infty} \int f_n.$$

Since this holds for all such bounded $g \leq \liminf_n f_n$ with $m(\text{supp } g) < \infty$, taking the supremum over g :

$$\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n. \quad \square$$

Theorem 5.2: Monotone Convergence Theorem

Proof. Since $0 \leq f_k \nearrow f$, monotonicity of the integral gives $\int f_k \leq \int f$ for all k , and the sequence $\{\int f_k\}$ is non-decreasing and bounded above by $\int f$, so $L := \lim_k \int f_k \leq \int f$ exists in $[0, \infty]$.

The reverse inequality follows by applying Fatou's Lemma to $\{f_k\}$:

$$\int f = \int \lim_k f_k = \int \liminf_{k \rightarrow \infty} f_k \leq \liminf_{k \rightarrow \infty} \int f_k = \lim_{k \rightarrow \infty} \int f_k = L.$$

Hence $L = \int f$. □

Theorem 5.3: Dominated Convergence Theorem

Proof. Since $|f_n| \leq g$ and $f_n \rightarrow f$ a.e., we have $|f| \leq g$, and $|f_n - f| \leq |f_n| + |f| \leq 2g$. Thus $2g - |f_n - f| \geq 0$.

Apply Fatou's Lemma to $2g - |f_n - f| \geq 0$ (using $f_n \rightarrow f$ a.e.):

$$\int 2g \leq \liminf_{n \rightarrow \infty} \int (2g - |f_n - f|) = \int 2g - \limsup_{n \rightarrow \infty} \int |f_n - f|.$$

Subtracting $\int 2g < \infty$ from both sides:

$$0 \leq -\limsup_{n \rightarrow \infty} \int |f_n - f|, \quad \text{i.e.} \quad \limsup_{n \rightarrow \infty} \int |f_n - f| \leq 0.$$

Since $\int |f_n - f| \geq 0$, we conclude $\lim_{n \rightarrow \infty} \int |f_n - f| = 0$. □

Proposition 7.1: Concentration of Integrability

Proof. Define $E_N := \{|x| < N\}$. The functions $|f|\chi_{E_N}$ are measurable and increase pointwise to $|f|$. By MCT:

$$\lim_{N \rightarrow \infty} \int |f|\chi_{E_N} = \int |f| < \infty.$$

Hence there exists N such that $\int_{E_N^c} |f| = \int |f| - \int_{E_N} |f| < \varepsilon$. Take $E = E_N$. □

Proposition 7.2: Absolute Continuity of the Integral

Proof. Suppose for contradiction no such δ exists. Then for some $\varepsilon > 0$ and every $n \in \mathbb{N}$, there is a measurable set E_n with $m(E_n) < 1/2^n$ yet $\int_{E_n} |f| \geq \varepsilon$.

Since $\sum_n m(E_n) < \infty$, the Borel–Cantelli Lemma gives $m(\limsup_n E_n) = 0$, so $\int_{\limsup E_n} |f| = 0$.

The functions $|f|\chi_{\bigcup_{k \geq n} E_k}$ are dominated by the integrable function $|f|$ and decrease to $|f|\chi_{\limsup E_n}$ pointwise. By DCT:

$$\lim_{n \rightarrow \infty} \int_{\bigcup_{k \geq n} E_k} |f| = \int_{\limsup E_n} |f| = 0.$$

But $E_n \subset \bigcup_{k \geq n} E_k$, so $\int_{E_n} |f| \leq \int_{\bigcup_{k \geq n} E_k} |f| \rightarrow 0$, contradicting $\int_{E_n} |f| \geq \varepsilon$. □