Real Analysis II: Study Note

Chapter 3: Basics of L^p Spaces

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1 The L^p Norm and L^p Spaces

Integration theory hands us a natural way to measure the "size" of a function: integrate its absolute value. But $L^1(\mathbb{R}^d)$ is only one point on a spectrum. Raising $|f|$ to the power $p \geq 1$ before integrating yields a family of spaces $L^p(\mathbb{R}^d)$ that trade off how finely they penalize large values versus spread-out mass. The goal of this chapter is to equip these spaces with a norm, verify that it is indeed a norm (the triangle inequality being the non-trivial part: Minkowski's inequality), and establish that these normed spaces are complete — making them **Banach spaces**, the workhorses of functional analysis. The chapter focuses primarily on L^1.

Roadmap: define L^p norm $\to$ Riesz–Fischer (completeness) $\to$ dense subsets $\to$ translation continuity $\to$ Hölder's inequality.

Definition 1.1: L^p Norm

For $1 \leq p < \infty$, the L^p **norm** of a measurable function f is

$$\|f\|_p \;:=\; \left(\int |f|^p \, \mathrm{d}m\right)^{1/p}.$$

For $p = \infty$, the L^∞ **norm** (essential supremum) is

$$\|f\|_\infty \;:=\; \inf\big\{M \geq 0 : |f(x)| \leq M \text{ a.e.}\big\}.$$

Checking the norm axioms.

(i) **Homogeneity.** $\|\lambda f\|_p = |\lambda| \, \|f\|_p$. *Yes*, by the $1/p$ exponent.

(ii) **Definiteness.** $\|f\|_p = 0 \Leftrightarrow f = 0$ a.e. This holds in L^p since elements are equivalence classes: $\|[f]\|_p = 0$ forces $f = 0$ a.e., i.e. $[f] = [0]$. Pointwise equality is not required.

(iii) **Triangle inequality.** $\|f + g\|_p \leq \|f\|_p + \|g\|_p$. *Yes*: this is Minkowski's inequality.

Failure of (ii) forces us to identify functions that agree a.e. Define the equivalence relation:

$$f \sim g \quad \text{iff} \quad f = g \text{ a.e.}$$

Definition 1.2: L^p Space

For $1 \leq p \leq \infty$,

$$L^p(\mathbb{R}^d) \;:=\; \big\{f : \mathbb{R}^d \to \mathbb{C} \;\big|\; f \text{ measurable},\ \|f\|_p < \infty\big\} \Big/ \sim$$

Elements of $L^p(\mathbb{R}^d)$ are **equivalence classes** $[f] := \{g : g \sim f\}$.

Remark 1.1. *L^p is a set of equivalence classes, not functions. In practice: work with a representative function and keep $\sim$ in mind. Within each class $[f]$, there is at most one continuous representative.*

Proposition 1.3: Properties of $L^1(\mathbb{R}^d)$

(i) $L^1(\mathbb{R}^d)$ is a **normed vector space** with:

- Norm: $\|f\|_1 := \int |f|$
- Metric: $d(f, g) := \|f - g\|_1$
- Convergence: $\int |f_n - f| \to 0$, given by DCT when $|f_n| \leq g$ integrable.

(ii) By Markov's inequality, $f_n \to f$ in L^1 implies $f_n \to f$ in measure[a]: for any $\varepsilon > 0$, $m(\{|f_n - f| > \varepsilon\}) \leq \frac{1}{\varepsilon}\|f_n - f\|_1 \to 0$. The converse fails in general.

(iii) **Simple functions are dense in** $L^1(\mathbb{R}^d)$: for every $f \in L^1$, there exist simple $\varphi_n \to f$ with $|\varphi_n| \leq |f|$.

Corollary: Step functions are also dense in $L^1(\mathbb{R}^d)$.

[a]In probability: L^1 convergence (convergence in mean) implies convergence in probability.

2 Completeness: The Riesz–Fischer Theorem

A normed space where Cauchy sequences always converge is called a **Banach space**. Whether a space is Banach determines whether one can safely pass to limits in norm, use series expansions, and apply fixed-point theorems. For L^1 with the Riemann integral, completeness fails: a Cauchy sequence of Riemann-integrable functions can converge to a non-Riemann-integrable limit. The Lebesgue integral repairs this.

Theorem 2.1: Riesz–Fischer Theorem[1]

$L^1(\mathbb{R}^d)$ is complete in the metric $d(f, g) = \|f - g\|_1$, hence a Banach space.

Remark 2.1. *(i) $L^1(\mathbb{R}^d)$ with the Riemann integral is* not *complete: a Cauchy sequence of Riemann-integrable functions can converge in L^1 to a function that is not Riemann-integrable. (ii) A **Banach space** is a* normed *vector space that is complete in the metric induced by its norm. (A complete metric space without a norm is not called a Banach space.) (iii) This completeness result holds for all $1 \leq p \leq \infty$.*

The following criterion replaces checking Cauchy sequences directly.

[1]The theorem holds for all $L^p(\mathbb{R}^d)$ with $p \geq 1$. In particular, L^2 is a *Hilbert* space (complete inner product space), the setting for Fourier analysis and quantum mechanics.

Theorem 2.2: Absolute Convergence Criterion for Completeness

A normed vector space is complete if and only if every absolutely convergent series converges:

$$\sum_{n=1}^{\infty} \|u_n\| < \infty \implies \lim_{n\to\infty} \left\| f - \sum_{k=1}^{n} u_k \right\| = 0 \quad \text{for some } f \text{ in the space.}$$

Proposition 2.3: Subsequence from L^1 Convergence

If $f_n \to f$ in L^1, then there exists a subsequence $\{f_{n_k}\}$ that converges to f pointwise a.e.

3 Dense Subsets of $L^1(\mathbb{R}^d)$

Knowing that simple and step functions are dense in L^1 is useful, but step functions have sharp edges that can be inconvenient. The key insight is that these edges can be smoothed: any indicator of a rectangle can be approximated in L^1 by a smooth function supported near the same rectangle.

Roadmap: step functions → smooth step approximation → $C_c^\infty(\mathbb{R}^d)$ is dense.

Approximating χ_{I_j} by smooth functions. Write $\chi_R(x) = \prod_{j=1}^{d} \chi_{I_j}(x_j)$ for a rectangle $R = \prod_j I_j$.

We want to control χ_{I_j} by an L^1 limit. Define a smoothed version by rounding the corners of the step, keeping $\|\psi - \chi_{I_j}\|_1 \le 2\varepsilon$.

The bump function.

$$h(w) := \begin{cases} e^{-\frac{1}{1-|w|^2}} & |w| < 1 \\ 0 & \text{otherwise,} \end{cases} \qquad g(x) := \int_{-\infty}^{x} h(s)\,\mathrm{d}s.$$

After scaling, $g(x)$ smoothly approximates χ_{I_j}, with $\|\psi - \chi_{I_j}\|_1 \le 2\varepsilon$.

Definition 3.1: $C_c^\infty(\mathbb{R}^d)$

$C_c^\infty(\mathbb{R}^d)$ denotes the space of **smooth, compactly supported** functions on $\mathbb{R}^d$.

Proposition 3.2: C_c^∞ is Dense in L^1

$C_c^\infty(\mathbb{R}^d)$ is dense in $L^1(\mathbb{R}^d)$: for every $f \in L^1(\mathbb{R}^d)$ and $\varepsilon > 0$, there exists $\phi \in C_c^\infty(\mathbb{R}^d)$ with $\|f - \phi\|_1 < \varepsilon$.

Remark 3.1. *This density result is a powerful proof technique: to establish an L^1 inequality or identity for arbitrary $f \in L^1$, it often suffices to verify it for $\phi \in C_c^\infty$ and then extend to all of L^1 by approximation, using continuity of the relevant operations in the L^1 norm.*

4 Translation Continuity

For $f \in L^1(\mathbb{R}^d)$ and $v \in \mathbb{R}^d$, define the **translation**

$$f_v(x) \;:=\; f(x-v).$$

Since Lebesgue measure is translation-invariant, $\|f_v\|_1 = \|f\|_1$ for all v.

A natural question: is the map $v \mapsto f_v$ *continuous* in the L^1 topology? That is, given $\varepsilon > 0$, does there exist $\delta > 0$ such that $|v| < \delta \Rightarrow \|f_v - f\|_1 < \varepsilon$?

By linearity, it suffices to check continuity at $v = 0$. But $f(x-v) \to f(x)$ pointwise as $v \to 0$ only if f is continuous at x — which can fail on a large set for a general $f \in L^1$. The fix is to first approximate f by $\phi \in C_c^\infty(\mathbb{R}^d)$, where pointwise continuity is guaranteed, and then transfer the estimate.

Proposition 4.1: Translation Continuity in L^1

For $f \in L^1(\mathbb{R}^d)$ and $v \in \mathbb{R}^d$, $\|f_v - f\|_1 \to 0$ as $|v| \to 0$.

5 Hölder's Inequality

The L^p spaces interact through a fundamental inequality. Just as the Cauchy–Schwarz inequality controls products in L^2, Hölder's inequality controls products across dual exponents, and is the key tool for everything from functional analysis to PDEs to probability.

Theorem 5.1: Hölder's Inequality[2]

Let $(X, \mathcal{F}, \mu)$ be a measure space, and let $p, q \in [1, \infty]$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$ (Hölder conjugates, where $1/\infty := 0$). If $f \in L^p$ and $g \in L^q$, then $fg \in L^1$ and

$$\|fg\|_1 \;\le\; \|f\|_p \, \|g\|_q.$$

Remark 5.1. *The pair (p, q) with $1/p + 1/q = 1$ are called **Hölder conjugates**. For $1 < p < \infty$, the dual exponent is $q = p/(p-1)$. The endpoint cases are $(p, q) = (1, \infty)$ and $(p, q) = (\infty, 1)$.*

Hölder's inequality is the engine behind Minkowski's inequality (L^p triangle inequality) and, on probability spaces, the inclusion $L^p \subset L^r$ for $r \le p$: if $\mathbb{E}[|X|^p] < \infty$ then $\mathbb{E}[|X|^r] \le \big(\mathbb{E}[|X|^p]\big)^{r/p} < \infty$, so higher moments dominate lower ones.

[2]The special case $p = q = 2$ is the **Cauchy–Schwarz inequality**: $\|fg\|_1 \le \|f\|_2 \|g\|_2$. In probability: $|\mathbb{E}[XY]| \le \sqrt{\mathbb{E}[X^2]\,\mathbb{E}[Y^2]}$. The endpoint cases $p = 1, q = \infty$ give $\|fg\|_1 \le \|f\|_1 \|g\|_\infty$, which is immediate from the definition of $\|\cdot\|_\infty$.

Proofs of Key Results

Theorem 2.1: Riesz–Fischer Theorem

Proof. We use Theorem 2.2. Let $\{f_n\} \subset L^1(\mathbb{R}^d)$ with $\sum_{n=1}^{\infty} \|f_n\|_1 < \infty$. Define

$$F(x) \;:=\; \sum_{n=1}^{\infty} |f_n(x)|.$$

By MCT (applying to the partial sums, which increase):

$$\int F \;=\; \sum_{n=1}^{\infty} \int |f_n| \;=\; \sum_{n=1}^{\infty} \|f_n\|_1 \;<\; \infty.$$

So $F \in L^1(\mathbb{R}^d)$ and $F(x) < \infty$ a.e. On the a.e. set where $F < \infty$, the series $\sum_n f_n(x)$ converges absolutely; define $f(x) := \sum_{n=1}^{\infty} f_n(x)$ there (and $f = 0$ elsewhere). Then f is measurable and $|f| \leq F \in L^1$, so $f \in L^1(\mathbb{R}^d)$.

It remains to show that the partial sums $S_N := \sum_{n=1}^{N} f_n$ converge to f in L^1. We have $|f - S_N| = \left|\sum_{n>N} f_n\right| \leq \sum_{n>N} |f_n| \leq F \in L^1$, and $f - S_N \to 0$ a.e. By DCT, $\|f - S_N\|_1 \to 0$. □

Proposition 2.3: Subsequence from L^1 convergence

Proof. If $f_n \to f$ in L^1, choose a subsequence $\{f_{n_k}\}$ with $\|f_{n_k} - f\|_1 < 1/2^k$. Set $g_k := f_{n_k} - f$; then $\sum_k \|g_k\|_1 < \infty$. By the Riesz–Fischer argument applied to $|g_k|$, $G(x) := \sum_k |g_k(x)|$ is finite a.e., so $\sum_k g_k(x)$ converges absolutely a.e. In particular, $g_k(x) \to 0$ a.e., i.e. $f_{n_k}(x) = f(x) + g_k(x) \to f(x)$ a.e. □

Theorem 5.1: Hölder's Inequality

Proof. If $\|f\|_p = \infty$ or $\|g\|_q = \infty$, the right-hand side is infinite and the inequality is trivial. If $\|f\|_p = 0$ or $\|g\|_q = 0$, then $f = 0$ a.e. or $g = 0$ a.e., so $fg = 0$ a.e. and $\|fg\|_1 = 0$.

Now assume $0 < \|f\|_p, \|g\|_q < \infty$. Normalise: let $\widetilde{f} = f/\|f\|_p$ and $\widetilde{g} = g/\|g\|_q$, so $\|\widetilde{f}\|_p = \|\widetilde{g}\|_q = 1$.

Apply **Young's inequality**: for $a, b \geq 0$ and $1/p + 1/q = 1$,

$$ab \;\leq\; \frac{a^p}{p} + \frac{b^q}{q},$$

(which follows from the concavity of log). Set $a = |\widetilde{f}(x)|$, $b = |\widetilde{g}(x)|$ and integrate:

$$\int |\widetilde{f}\widetilde{g}| \;\leq\; \frac{1}{p}\int |\widetilde{f}|^p + \frac{1}{q}\int |\widetilde{g}|^q \;=\; \frac{1}{p} + \frac{1}{q} \;=\; 1.$$

Rescaling: $\int |fg| \leq \|f\|_p \|g\|_q$, i.e. $\|fg\|_1 \leq \|f\|_p \|g\|_q$. □