Real Analysis II: Study Note

Chapter 4: Fubini's Theorem

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1 Multiple Integrals and the Problem of Iteration

In calculus, a double integral over $\mathbb{R}^2$ is computed by iterating two one-dimensional integrals — first integrating over x, then over y, or vice versa. This works cleanly for Riemann integration of continuous functions. The Lebesgue setting raises a sharper question: for a general $f \in L^1(\mathbb{R}^d)$, can we always swap the order of integration?

Roadmap: counterexample showing order can matter → Fubini's theorem (for L^1 functions) → Tonelli's theorem (for non-negative functions) → proof strategy via 6 claims.

The central question is:

$$\int_{\mathbb{R}^d} f \;=\; \iint f(x,y)\,\mathrm{d}x\,\mathrm{d}y \;\stackrel{?}{=}\; \int\left(\int f(x,y)\,\mathrm{d}y\right)\mathrm{d}x \;=\; \int\left(\int f(x,y)\,\mathrm{d}x\right)\mathrm{d}y.$$

Counterexample: order of integration can matter.

Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined on unit squares near the axes, arranged so that each row-slice sums to $+1$ and each column-slice sums to 0:

$$\int\left[\int f\,\mathrm{d}y\right]\mathrm{d}x = 1+0+1+\cdots \neq 0 = 0+0+\cdots = \int\left[\int f\,\mathrm{d}x\right]\mathrm{d}y,$$

and the double integral $\int_{\mathbb{R}^2} f$ does not exist. The culprit: $f \notin L^1(\mathbb{R}^2)$, so $\int |f| = \infty$. Fubini's theorem shows that L^1 integrability is the right sufficient condition to rule out such pathologies.

2 Fubini's Theorem

Theorem 2.1: Fubini's Theorem[1]

Let $f \in L^1(\mathbb{R}^d)$ where $d = d_1 + d_2$, so that $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and points are written (x,y) with $x \in \mathbb{R}^{d_1}$, $y \in \mathbb{R}^{d_2}$. Then:

(i) The **slice** $f(\,\cdot\,,y) : x \mapsto f(x,y)$ is integrable over $\mathbb{R}^{d_1}$ for *almost every* $y \in \mathbb{R}^{d_2}$.

(ii) The **integrated slice** $y \mapsto \displaystyle\int_{\mathbb{R}^{d_1}} f(x,y)\,\mathrm{d}x$ is integrable over $\mathbb{R}^{d_2}$ (defined for a.e. y).

(iii) The **iterated integral equals the full integral**:

$$\int_{\mathbb{R}^d} f \;=\; \int_{\mathbb{R}^{d_2}}\left(\int_{\mathbb{R}^{d_1}} f(x,y)\,\mathrm{d}x\right)\mathrm{d}y.$$

[1] By symmetry the same holds with x and y swapped, so in particular $\int_{\mathbb{R}^{d_1}}\left(\int_{\mathbb{R}^{d_2}} f\,\mathrm{d}y\right)\mathrm{d}x = \int_{\mathbb{R}^{d_2}}\left(\int_{\mathbb{R}^{d_1}} f\,\mathrm{d}x\right)\mathrm{d}y$. The proof is given in the Proofs section and proceeds via 6 structural claims.

3 Tonelli's Theorem

Fubini requires $f \in L^1$, which presupposes $\int |f| < \infty$. But verifying integrability often requires first computing an iterated integral. Tonelli's theorem breaks this circularity for non-negative functions: for $f \geq 0$, the iterated integral always equals the full integral (possibly $+\infty$), so one can use the iterated integral to *check* integrability before applying Fubini. The standard workflow is: apply Tonelli to $|f|$ first; if the iterated integral is finite, conclude $f \in L^1$ and then apply Fubini to f.

Theorem 3.1: Tonelli's Theorem[2]

For $f \geq 0$ measurable on $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, the conclusions of Fubini's Theorem hold (with values possibly $+\infty$):

$$\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y = \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x.$$

In particular, if either iterated integral is finite, then $f \in L^1(\mathbb{R}^d)$ and Fubini's Theorem applies.

Proof sketch. Choose $f_n \in L^1$, $f_n \geq 0$, $f_n \nearrow f$ (e.g. $f_n = \min(f,n) \cdot \chi_{\{|x|<n\}}$). By Fubini applied to each f_n:

$$\int_{\mathbb{R}^d} f_n = \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f_n\,\mathrm{d}x \right) \mathrm{d}y.$$

Apply MCT as $n \to \infty$: on the left $\int f_n \nearrow \int f$; on the right, the inner integral $\int f_n\,\mathrm{d}x \nearrow \int f\,\mathrm{d}x$ for a.e. y (inner MCT), then the outer integral increases as well (outer MCT). Both sides converge to the same limit.

Corollary 3.2: Fubini–Tonelli Workflow

If f is measurable and $\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} |f(x,y)|\,\mathrm{d}x \right) \mathrm{d}y < \infty$, then $f \in L^1(\mathbb{R}^d)$ and Fubini's Theorem applies to f.

Remark 3.1. *Fubini's theorem is stated for Lebesgue measure on $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. The result* does *extend to general σ-finite*[3] *product measure spaces $(X \times Y, \mu \times \nu)$: if μ and ν are both σ-finite, then Fubini's and Tonelli's theorems hold for $(X \times Y, \mu \times \nu)$. Without σ-finiteness both theorems can fail.*

[2] Tonelli's theorem is the non-negative precursor to Fubini's theorem. Together they are often used in tandem: first apply Tonelli to $|f|$ to verify $\int |f| < \infty$, then apply Fubini to f to exchange the order of integration. Proof strategy: approximate $f \geq 0$ by $f_n \in L^1$ with $f_n \nearrow f$, $f_n \geq 0$. Apply Fubini to each f_n, then apply MCT twice.

[3] A measure space $(X, \mathcal{F}, \mu)$ is **σ-finite** if $X = \bigcup_{n=1}^{\infty} E_n$ with $\mu(E_n) < \infty$ for each n.

Proof of Fubini's Theorem

Let $\mathcal{F} := \{f \in L^1(\mathbb{R}^d) : \text{conclusions (i)–(iii) hold for } f\}$. We show $\mathcal{F} = L^1(\mathbb{R}^d)$ by establishing six claims.

The starting point: for any rectangle $R = R_1 \times R_2$ (with $R_k \subset \mathbb{R}^{d_k}$), $\chi_R \in \mathcal{F}$ trivially, so $\mathcal{F}$ is non-empty.

Claim I: $\mathcal{F}$ is closed under linear combinations

Proof. $L^1(\mathbb{R}^d)$ is a vector space, and all three conclusions (i)–(iii) are linear in f. □

Claim II: $\mathcal{F}$ is closed under monotone limits

Proof. Suppose $f_k \in \mathcal{F}$ with $f_k \nearrow f$ and $f \in L^1(\mathbb{R}^d)$; the decreasing case $f_k \searrow f$ follows by applying the increasing case to $f_1 - f_k \nearrow f_1 - f$ (valid since $m(f_1) < \infty$ ensures $f_1 - f_k \geq 0$).

By hypothesis (i) for each f_k: $f_k(\cdot, y)$ is integrable over $\mathbb{R}^{d_1}$ for a.e. y, on a full-measure set $A_k \subset \mathbb{R}^{d_2}$. Set $A = \bigcap_k A_k$; then $m(A^c) = 0$.

For $y \in A$, $f_k(\cdot, y) \nearrow f(\cdot, y)$ pointwise, so by MCT: $g_k(y) := \int_{\mathbb{R}^{d_1}} f_k(x, y)\,\mathrm{d}x \nearrow g(y) := \int_{\mathbb{R}^{d_1}} f(x, y)\,\mathrm{d}x$.

By hypothesis (ii) each g_k is integrable over $\mathbb{R}^{d_2}$, and $g_k \nearrow g$. By hypothesis (iii): $\int_{\mathbb{R}^d} f_k = \int_{\mathbb{R}^{d_2}} g_k$. Since $f \in L^1$, $\int_{\mathbb{R}^d} f_k \nearrow \int_{\mathbb{R}^d} f < \infty$; applying MCT to $g_k \nearrow g$ over $\mathbb{R}^{d_2}$:

$$\int_{\mathbb{R}^{d_2}} g = \lim_k \int_{\mathbb{R}^{d_2}} g_k = \lim_k \int_{\mathbb{R}^d} f_k = \int_{\mathbb{R}^d} f < \infty.$$

Thus $g \in L^1(\mathbb{R}^{d_2})$ (conclusion (ii)) and $\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^{d_2}} g$ (conclusion (iii)). Since $g < \infty$ a.e., $f(\cdot, y)$ is integrable for a.e. y (conclusion (i)). □

Claim III: $\chi_G \in \mathcal{F}$ for any open $G \subset \mathbb{R}^d$ with $m(G) < \infty$, and for any G_δ set with finite measure

Proof. For any open set G, write $G = \bigcup_{j \in \mathbb{N}} R_j$ (countable union of almost disjoint rectangles, by Theorem 1.4 from Chapter 1). The partial unions $\chi_{U_n} := \chi_{\bigcup_{j=1}^n R_j} \nearrow \chi_G$. Since $\chi_{R_j} \in \mathcal{F}$ and $\mathcal{F}$ is closed under linear combinations (Claim I), each $\chi_{U_n} \in \mathcal{F}$. By Claim II, $\chi_G \in \mathcal{F}$ (provided $m(G) < \infty$).

For a G_δ set $G = \bigcap_{k \in \mathbb{N}} U_k$ (open U_k), take partial intersections $\chi_{G_n} = \chi_{\bigcap_{k=1}^n U_k} \searrow \chi_G$; Claim II gives $\chi_G \in \mathcal{F}$. □

Claim IV: $\chi_E \in \mathcal{F}$ for any $E \subset \mathbb{R}^d$ with $m(E) = 0$

Proof. Approximate E by a G_δ set $G \supset E$ with $m(G) = 0$ (possible since $m(E) = 0$). By Claim III, $\chi_G \in \mathcal{F}$ and $\int_{\mathbb{R}^d} \chi_G = 0$. Since $\chi_G \geq 0$ and its full integral is 0, for a.e. y the inner integral $\int_{\mathbb{R}^{d_1}} \chi_G(x, y)\, \mathrm{d}x = 0$, i.e. the x-slice of G at height y has measure 0.

Since $E \subset G$, the x-slice of E also has outer measure 0 (by monotonicity), so $\chi_E(\cdot, y)$ is measurable and $\int \chi_E(\cdot, y)\, \mathrm{d}x = 0$ for a.e. y. This gives all three conclusions for χ_E. □

Remark 3.2. *In Claim IV, we can only use outer measure for the slices of E, since slices of Lebesgue measurable sets need not be Lebesgue measurable (they are for Borel sets). Outer measure monotonicity still suffices.*

Claim V: $\chi_E \in \mathcal{F}$ for any measurable E with $m(E) < \infty$

Proof. By Borel approximation (Proposition from Chapter 1): there exists a G_δ set $B \supset E$ with $m(B \setminus E) = 0$. By Claim III, $\chi_B \in \mathcal{F}$. By Claim IV, $\chi_{B \setminus E} \in \mathcal{F}$ (since $m(B \setminus E) = 0$). Since $\chi_E = \chi_B - \chi_{B \setminus E}$, Claim I gives $\chi_E \in \mathcal{F}$. □

Claim VI: $f \in \mathcal{F}$ for all $f \in L^1(\mathbb{R}^d)$

Proof. Write $f = f^+ - f^-$. By Claim I it suffices to show $f^+, f^- \in \mathcal{F}$. Since both are non-negative and in L^1, fix attention on $f^+ \geq 0$.

Let $\{\varphi_j\}$ be simple functions with $\varphi_j \nearrow f^+$ (such exist by the Simple Function Approximation Theorem). Each $\varphi_j = \sum_k c_k \chi_{E_k}$ with $m(E_k) < \infty$, so $\varphi_j \in \mathcal{F}$ by Claims I and V. By Claim II, $f^+ \in \mathcal{F}$. Similarly $f^- \in \mathcal{F}$, and therefore $f \in \mathcal{F}$. □

Claims I–VI together show $L^1(\mathbb{R}^d) \subset \mathcal{F}$, completing the proof of Fubini's Theorem. ■

Remark 3.3. *The proof makes essential use of the structure of $\mathbb{R}^d$: rectangles, the open-set decomposition theorem, and Borel approximation. For a general product measure space $(X \times Y, \mu \times \nu)$, the same conclusion holds under σ-finiteness, but the rectangular structure of $\mathbb{R}^d$ is what makes the current proof work cleanly.*