Real Analysis II: Study Note

Chapter 5: Hilbert Space

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1 Inner Product Spaces

A normed space measures size. An inner product space additionally measures *angle*, enabling orthogonality and projection. The payoff is dramatic: every separable Hilbert space has an orthonormal basis, and every L^2 function admits a unique expansion in that basis (Parseval's identity). The chapter culminates in Fourier series as the concrete orthonormal basis for $L^2([0, 2\pi])$.

Roadmap: inner product → Hilbert space → ONB & Bessel → Parseval → Fourier series → Fourier transform → Schwartz functions.

Definition 1.1: Inner Product

On a complex vector space V, an **inner product** $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ satisfies:

(i) **Sesquilinearity:** $\langle v, w\rangle$ is linear in v and conjugate-linear in w:

$$\langle cv, w\rangle = c\langle v, w\rangle, \qquad \langle v, cw\rangle = \overline{c}\langle v, w\rangle.$$

(ii) **Conjugate symmetry:** $\langle v, w\rangle = \overline{\langle w, v\rangle}$.

(iii) **Positive definiteness:** $\langle v, v\rangle \geq 0$, with equality iff $v = 0$.

An inner product induces a norm $\|v\| := \sqrt{\langle v, v\rangle}$ and hence a metric.

Theorem 1.2: Cauchy–Schwarz Inequality

For all v, w in an inner product space,

$$|\langle v, w\rangle| \ \leq \ \|v\| \, \|w\|.$$

Proof idea: For any $a \in \mathbb{C}$, expand $0 \leq \|v - aw\|^2 = \|v\|^2 - a\langle w, v\rangle - \overline{a}\langle v, w\rangle + |a|^2\|w\|^2$. Choose $a = \langle v, w\rangle / \|w\|^2$ (when $w \neq 0$) and simplify to obtain $|\langle v, w\rangle|^2 \leq \|v\|^2\|w\|^2$.

Examples.

(1) **Euclidean $\mathbb{R}^n$:** $\langle v, w\rangle := v \cdot w = \sum_j v_j w_j$.

(2) $L^2(E)$**:** $\langle f, g\rangle := \int_E f\overline{g}$.

(3) $L^2(X, \mathrm{d}\mu)$ for any measure space $(X, \mathcal{F}, \mu)$: $\langle f, g\rangle := \int_X f\overline{g}\,\mathrm{d}\mu$.

Definition 1.3: Hilbert Space

A **Hilbert space** H is a complete inner product space (complete in the metric induced by the norm $\|v\| = \sqrt{\langle v, v\rangle}$).

Remark 1.1. *For finite-dimensional normed spaces, completeness is automatic. In infinite dimensions it must be checked. $L^2(\mathbb{R}^d)$ is complete by the Riesz–Fischer theorem (Chapter 3).*

2 Orthonormal Bases

Definition 2.1: Separable Hilbert Space & Orthonormality

H is **separable** if it has a countable dense subset (e.g., in L^2: step functions with rational coefficients). A sequence $\{e_j\}$ is **orthonormal** if

$$\langle e_i, e_j \rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

$\{e_j\}$ is an **orthonormal basis (ONB)** if $\overline{\operatorname{span}\{e_j\}} = H$ (finite linear combinations are dense).

Fourier coefficients. Given $v \in H$ and an orthonormal sequence $\{e_j\}$, define $a_j := \langle v, e_j \rangle$ and partial sums $S_N := \sum_{j=1}^{N} a_j e_j$.

Theorem 2.2: Bessel's Inequality

For any $v \in H$ and orthonormal sequence $\{e_j\}$,

$$\sum_{j=1}^{\infty} |\langle v, e_j \rangle|^2 \;\leq\; \|v\|^2.$$

In particular $\|S_N\|^2 = \sum_{j=1}^{N} |\langle v, e_j \rangle|^2 \leq \|v\|^2$ for all N. Equality holds for all $v \in H$ if and only if $\{e_j\}$ is an ONB (Parseval's identity).

Proposition 2.3: Convergence of Fourier Series

If $\{e_j\}$ is orthonormal in H and $\sum_j |a_j|^2 < \infty$, then $\sum_j a_j e_j$ converges in H. Indeed, by the Pythagorean theorem, for $M > N$: $\|S_M - S_N\|^2 = \sum_{j=N+1}^{M} |a_j|^2 \to 0$ as $N \to \infty$, so $\{S_N\}$ is Cauchy in H, hence convergent by completeness.

Theorem 2.4: Basis Expansion Theorem (TFAE)

Let $\{e_j\}$ be orthonormal in H. The following are equivalent:

(i) $\{e_j\}$ is an ONB (i.e., $\overline{\operatorname{span}\{e_j\}} = H$).

(ii) $\langle v, e_j \rangle = 0$ for all j implies $v = 0$ (**completeness**).

(iii) $S_N \to v$ in H for every $v \in H$ (i.e., $v = \sum_j \langle v, e_j \rangle e_j$).

(iv) **Parseval's Identity:** $\displaystyle\sum_j |\langle v, e_j \rangle|^2 = \|v\|^2$ for every $v \in H$.

The polarised form of (iv) also holds: $\langle v, w \rangle = \sum_j \langle v, e_j \rangle \overline{\langle w, e_j \rangle}$ for all $v, w \in H$.

Theorem 2.5: Existence of ONB

Every separable Hilbert space admits an orthonormal basis.

Proof: Iteratively apply the Gram–Schmidt process to a countable dense subset.

3 Fourier Series

A function $f \in L^2([0, 2\pi])$ admits an L^2-convergent expansion in trigonometric functions:

$$f = \sum_{k\in\mathbb{Z}} a_k e^{ik\,\cdot} \quad \text{in } L^2([0, 2\pi]), \qquad a_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-ikx}\,\mathrm{d}x.$$

Pointwise convergence is a separate (harder) question and requires additional regularity on f.

Connection to the heat equation. On $[0, 2\pi]$ with periodic boundary conditions, the heat equation $\partial_t u = \partial_{xx} u$ with $u(x, 0) = f(x)$ is diagonalised by Fourier modes. Setting $u(x, t) = \sum_k a_k(t)e^{ikx}$:

$$\partial_t(a_k e^{ikx}) = \partial_{xx}(a_k e^{ikx}) = -k^2 a_k e^{ikx} \;\Rightarrow\; \partial_t a_k = -k^2 a_k \;\Rightarrow\; a_k(t) = a_k(0)\, e^{-k^2 t}.$$

So the solution is $u(x, t) = \sum_k a_k(0)e^{-k^2 t}e^{ikx}$, which decays rapidly since the high-frequency modes are damped by $e^{-k^2 t}$.

Theorem 3.1: Fourier Series as ONB for $L^2([0, 2\pi])$

The functions $e_k(\theta) := \frac{1}{\sqrt{2\pi}} e^{ik\theta}$, $k \in \mathbb{Z}$, form an orthonormal basis for $L^2([0, 2\pi])$.

Strategy (Poisson kernel). Solve $\Delta u = 0$ on the unit disk $\mathbb{D} = \{(r, \theta) : 0 \le r < 1\}$ with boundary condition $u|_{\partial\mathbb{D}} = f$ (where f is a 2π-periodic function on the circle). The solution via the Poisson kernel is:

$$P_r(\theta) = \frac{1}{2\pi} \cdot \frac{1 - r^2}{1 - 2r\cos\theta + r^2}, \qquad u(r, \theta) = (P_r * f)(\theta) = \int_0^{2\pi} P_r(\theta - \varphi) f(\varphi)\,\mathrm{d}\varphi,$$

where $*$ denotes circular convolution on $[0, 2\pi]$.

Theorem 3.2: Poisson Kernel Convergence

For $f \in L^1([0, 2\pi])$, $P_r * f \xrightarrow{r\to 1^-} f$ a.e.

Example. Let $f(\theta) = \chi_{[0,\pi]}$, and $e_k(\theta) = \frac{1}{\sqrt{2\pi}} e^{ik\theta}$ (consistent with Theorem 3.1).

Fourier coefficients $a_k = \langle f, e_k \rangle = \int_0^{2\pi} f(\theta)\overline{e_k(\theta)}\,\mathrm{d}\theta$:

$$a_0 = \frac{1}{\sqrt{2\pi}} \int_0^{\pi} 1\,\mathrm{d}\theta = \sqrt{\frac{\pi}{2}}, \qquad a_k = \frac{1}{\sqrt{2\pi}} \int_0^{\pi} e^{-ik\theta}\,\mathrm{d}\theta = \frac{1 - e^{-ik\pi}}{\sqrt{2\pi}\,ik} = \begin{cases} \dfrac{2}{\sqrt{2\pi}\,ik} & k \text{ odd}, \\ 0 & k \text{ even}, k \neq 0. \end{cases}$$

$\|f\|_{L^2}^2 = \int_0^{\pi} 1\,\mathrm{d}\theta = \pi$. **Check Parseval:**

$$\sum_{k\in\mathbb{Z}} |a_k|^2 = |a_0|^2 + \sum_{\substack{k\in\mathbb{Z} \\ k \text{ odd}}} |a_k|^2 = \frac{\pi}{2} + 2 \sum_{k=1,3,5,\ldots} \frac{4}{2\pi k^2} = \frac{\pi}{2} + \frac{4}{\pi} \cdot \frac{\pi^2}{8} = \pi = \|f\|_{L^2}^2. \checkmark$$

(This also yields $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$.*)*

4 Fourier Transform

Definition 4.1: Fourier Transform on L^1

For $f \in L^1(\mathbb{R}^d)$, define $\widehat{f} : \mathbb{R}^d \to \mathbb{C}$ by

$$\widehat{f}(\xi) \;:=\; \int_{\mathbb{R}^d} \exp(-2\pi i\, x \cdot \xi)\, f(x)\,\mathrm{d}x, \qquad \xi \in \mathbb{R}^d.$$

Claim: The natural domain for the Fourier transform is L^2, even though $\widehat{f}$ is initially defined for $f \in L^1$.

Theorem 4.2: Riemann–Lebesgue Lemma

For $f \in L^1(\mathbb{R}^d)$: $\widehat{f}$ is bounded ($\|\widehat{f}\|_\infty \leq \|f\|_1$), uniformly continuous, and $\widehat{f}(\xi) \to 0$ as $|\xi| \to \infty$. In other words, $\widehat{f} \in C_0(\mathbb{R}^d)$ (continuous functions vanishing at infinity), but $\widehat{f}$ need not be in $L^1(\mathbb{R}^d)$.

Other conventions (no universally agreed sign/normalization):

$$\widehat{f} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f\,\mathrm{d}x, \qquad \widehat{f} = \int_{\mathbb{R}^d} e^{-ikx} f\,\mathrm{d}x.$$

Next steps: Extend the Fourier transform to L^2; use Schwartz functions as an intermediate step.

5 Schwartz Functions

Definition 5.1: Multi-index Notation

A **multi-index** $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $\alpha_j \in \{0, 1, 2, \ldots\}$ encodes: $|\alpha| = \sum_j \alpha_j$, $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, $\alpha! = \alpha_1! \cdots \alpha_d!$, and $D^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$.

Definition 5.2: Schwartz Space

The **Schwartz space** is

$$\mathcal{S}(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d) : \sup_{x \in \mathbb{R}^d} |x^\beta D^\alpha f(x)| < \infty \; \forall \, \text{multi-indices } \alpha, \beta \right\}.$$

These are smooth functions that *rapidly decay*: $D^\alpha f(x) \to 0$ faster than any polynomial as $|x| \to \infty$.

Examples: $C_c^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$; $e^{-|x|^2} \in \mathcal{S}$; $\exp\left(-x^\top A x\right) \in \mathcal{S}$ for any positive definite A.

Proposition 5.3

$C_c^\infty(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$ (in the Schwartz topology), and $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ for all $1 \le p < \infty$. (Density fails for $p = \infty$.)

Lemma 5.4: Fourier Transform on $\mathcal{S}$

If $f \in \mathcal{S}(\mathbb{R}^d)$, then $\widehat{f} \in \mathcal{S}(\mathbb{R}^d)$.

Proof idea (integration by parts):

$$\widehat{\partial_j f}(\xi) = 2\pi i \, \xi_j \, \widehat{f}(\xi), \qquad \widehat{x_j f}(\xi) = \frac{i}{2\pi} \partial_j \widehat{f}(\xi).$$

Mapping properties of $\mathcal{F}$:

$$x_j \cdot \;\longleftrightarrow\; \tfrac{i}{2\pi} \partial_{\xi_j}, \qquad \partial_{x_j} \;\longleftrightarrow\; 2\pi i \, \xi_j \cdot$$

$\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is a bijection, with $\langle \widehat{f}, g \rangle_{L^2} = \langle f, \widehat{g} \rangle_{L^2}$.

Theorem 5.5: Fourier Transform on L^2 (Plancherel)

The Fourier transform $\mathcal{F}$, initially defined on $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, extends uniquely to a unitary bijection $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ satisfying **Plancherel's theorem**:

$$\|\widehat{f}\|_{L^2} = \|f\|_{L^2} \qquad \text{for all } f \in L^2(\mathbb{R}^d).$$

More generally, $\langle \widehat{f}, \widehat{g} \rangle_{L^2} = \langle f, g \rangle_{L^2}$ for all $f, g \in L^2$. The inverse transform is $\mathcal{F}^{-1}[g](x) =$

$\int_{\mathbb{R}^d} e^{2\pi i\, x\cdot\xi}\, g(\xi)\,\mathrm{d}\xi$ (in the L^2 sense).