Real Analysis II: Study Note

Chapter 7: Product Measure

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Spring 2026

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1 Constructing the Product Measure

Chapter 4 proved Fubini's theorem for Lebesgue measure on $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, using the special rectangular structure of $\mathbb{R}^d$. This chapter asks: given two abstract measure spaces $(X_1, \mathcal{M}_1, \mu_1)$ and $(X_2, \mathcal{M}_2, \mu_2)$, can we always build a product measure $\mu_1 \times \mu_2$ on $X_1 \times X_2$ and apply Fubini? The answer requires σ-finiteness.

Roadmap: measurable rectangles → algebra $\mathcal{A}$ → premeasure μ_0 → product σ-algebra $\mathcal{M}_1 \otimes \mathcal{M}_2$ → σ-finiteness → Fubini → signed measures → Radon–Nikodym.

Definition 1.1: Measurable Rectangles and the Product σ-Algebra

Given $(X_1, \mathcal{M}_1, \mu_1)$ and $(X_2, \mathcal{M}_2, \mu_2)$:

- **Measurable rectangles:** sets of the form $A \times B$ with $A \in \mathcal{M}_1$, $B \in \mathcal{M}_2$.
- $\mathcal{A}$: the **algebra** generated by measurable rectangles (closed under complements and finite unions/intersections). Note: $\mathcal{A}$ is *not* itself a σ-algebra.
- **Product σ-algebra:** $\mathcal{M}_1 \otimes \mathcal{M}_2 := \sigma(\mathcal{A})$, the σ-algebra generated by all measurable rectangles.
- **Premeasure on $\mathcal{A}$:** $\mu_0(A \times B) := \mu_1(A)\mu_2(B)$.

Remark 1.1. *$\mathcal{M}_1 \times \mathcal{M}_2$ (the Cartesian product) is* not *the product σ-algebra. The product σ-algebra $\mathcal{M}_1 \otimes \mathcal{M}_2 = \sigma(\mathcal{A})$ is the correct object.*

Lemma 1.2

μ_0 is a premeasure on $\mathcal{A}$.

By the Carathéodory Extension Theorem (Chapter 8), μ_0 extends to a unique measure $\mu_1 \times \mu_2$ on $\mathcal{M}_1 \otimes \mathcal{M}_2$.

2 σ-Finiteness and Fubini's Theorem

Definition 2.1: σ-Finite Measure Space

$(X, \mathcal{M}, \mu)$ is **σ-finite** if there exist measurable sets $X_1, X_2, \ldots \in \mathcal{M}$ such that $X = \bigcup_{j=1}^{\infty} X_j$ and $\mu(X_j) < \infty$ for all j. Equivalently, the sets X_j may be taken to be pairwise disjoint.

Examples. Lebesgue measure on $\mathbb{R}^d$ is σ-finite ($\mathbb{R}^d = \bigcup_n B(0, n)$, each ball having finite measure). Counting measure on $\mathbb{N}$ is σ-finite ($\mathbb{N} = \bigcup_n \{n\}$, each singleton having measure 1). However, counting measure on $\mathbb{R}$ (or any uncountable set) is *not* σ-finite: any countable union of finite-measure sets in the counting measure has only countably many points, hence cannot cover $\mathbb{R}$.

Why σ-finiteness is necessary. Consider $X_1 = \mathbb{R}$ with Lebesgue measure m and $X_2 = \mathbb{R}$ with counting measure $\#$. Let $E = \{(x,x) : x \in [0,1]\}$ (the diagonal). Then:

$$\int\int \chi_E(x,y)\,\mathrm{d}m(x)\,\mathrm{d}\#(y) = \int_{\mathbb{R}} m(\{x : (x,y) \in E\})\,\mathrm{d}\#(y) = \int_{\mathbb{R}} 0\,\mathrm{d}\#(y) = 0,$$
$$\int\int \chi_E(x,y)\,\mathrm{d}\#(y)\,\mathrm{d}m(x) = \int_{\mathbb{R}} \#(\{y : (x,y) \in E\})\,\mathrm{d}m(x) = \int_{\mathbb{R}} 1\,\mathrm{d}m(x) = \infty.$$

Iterated integrals disagree because the counting measure is not σ-finite. σ-finiteness on both factors rules out this pathology.

Theorem 2.2: Fubini–Tonelli for Product Measures

Let $(X_1, \mathcal{M}_1, \mu_1)$ and $(X_2, \mathcal{M}_2, \mu_2)$ be σ-finite measure spaces.

(i) **(Tonelli)** If $f \geq 0$ is measurable on $X_1 \times X_2$, then

$$\int_{X_1 \times X_2} f\,\mathrm{d}(\mu_1 \times \mu_2) = \int_{X_1} \left(\int_{X_2} f(x,y)\,\mathrm{d}\mu_2(y) \right) \mathrm{d}\mu_1(x) = \int_{X_2} \left(\int_{X_1} f(x,y)\,\mathrm{d}\mu_1(x) \right) \mathrm{d}\mu_2(y),$$

with all quantities possibly equal to $+\infty$.

(ii) **(Fubini)** If $f \in L^1(X_1 \times X_2, \mu_1 \times \mu_2)$, then the same equalities hold with all quantities finite.

The proof follows the same 6-claim strategy as Chapter 4 (via premeasures and Carathéodory extension), using the lemmas below.

Lemma 2.3

For $E \in \mathcal{M}_1 \otimes \mathcal{M}_2$ with $(\mu_1 \times \mu_2)(E) = 0$: slices $E_x := \{y : (x,y) \in E\}$ satisfy $\mu_2(E_x) = 0$ for μ_1-a.e. x.

Lemma 2.4

For non-negative f in $L^1(X_1 \times X_2, \mathrm{d}\mu_1 \times \mathrm{d}\mu_2)$, Fubini holds.

3 Signed Measures

Definition 3.1: Signed Measure

A **signed measure** on $(X, \mathcal{M})$ is a countably additive function $\nu : \mathcal{M} \to (-\infty, \infty]$ (or $[-\infty, \infty)$) — finite on at least one side.

Example. Given $f \in L^1(X, \mathrm{d}\mu)$, define $\nu(A) := \int_A f\,\mathrm{d}\mu$. Then ν is a signed measure; if $f \geq 0$ it is a measure, otherwise a signed measure.

Theorem 3.2: Hahn Decomposition

For any signed measure ν on $(X, \mathcal{M})$, there exists a partition $X = P \sqcup N$ (the **Hahn decomposition**) where P is a **positive set** ($\nu(E) \geq 0$ for all $E \subset P$) and N is a **negative set** ($\nu(E) \leq 0$ for all $E \subset N$).

Proposition 3.3: Jordan Decomposition

Given a signed measure ν on $(X, \mathcal{M})$, define

$$\nu^+(E) := \nu(E \cap P), \qquad \nu^-(E) := -\nu(E \cap N),$$

using a Hahn decomposition $X = P \sqcup N$. Then ν^+, ν^- are positive measures with $\nu^+ \perp \nu^-$, and $\nu = \nu^+ - \nu^-$ (the **Jordan decomposition**). This decomposition is unique and minimal: if $\nu = \mu_1 - \mu_2$ with μ_1, μ_2 positive measures, then $\mu_1 \geq \nu^+$ and $\mu_2 \geq \nu^-$.
The **total variation measure** is $|\nu| := \nu^+ + \nu^-$, satisfying:

$$|\nu|(E) = \sup\left\{ \sum_j |\nu(E_j)| : E = \bigsqcup_j E_j \text{ measurable, disjoint} \right\}, \qquad |\nu(E)| \leq |\nu|(E).$$

4 Absolute Continuity and Singularity of Measures

Definition 4.1: Absolute Continuity and Mutual Singularity

Given two measures μ, ν on $(X, \mathcal{M})$:

(1) ν is **absolutely continuous** w.r.t. μ (written $\nu \ll \mu$) if

$$\mu(E) = 0 \;\Rightarrow\; \nu(E) = 0.$$

Example: If $\mathrm{d}\nu = f \,\mathrm{d}\mu$ for $f \geq 0$, then $\nu \ll \mu$.

(2) μ and ν are **mutually singular** (written $\mu \perp \nu$) if $X = A \sqcup B$ with $\mu(B) = 0$ and $\nu(A) = 0$ (they live on disjoint sets).

Example: The Dirac measure δ_p on $\mathbb{R}$ is singular w.r.t. Lebesgue measure: take $A = \{p\}$ and $B = \{p\}^c$; then $m(\{p\}) = 0$ (so μ cannot see A) and $\delta_p(\{p\}^c) = 0$ (so ν cannot see B).

5 Radon–Nikodym Theorem

Theorem 5.1: Lebesgue–Radon–Nikodym Theorem[1]

Let $(X, \mathcal{M})$ be a measurable space, μ a σ-finite positive measure, and ν a σ-finite signed measure. Then:

(i) **(Lebesgue Decomposition)** There exists a unique decomposition $\nu = \nu_a + \nu_s$ with $\nu_a \ll \mu$ and $\nu_s \perp \mu$.

(ii) **(Radon–Nikodym)** There exists a unique (up to μ-a.e. equivalence) measurable function $f : X \to [-\infty, \infty]$, with $f \geq 0$ when $\nu \geq 0$, such that

$$\mathrm{d}\nu_a = f \, \mathrm{d}\mu, \quad \text{i.e.,} \quad \nu_a(E) = \int_E f \, \mathrm{d}\mu \quad \forall E \in \mathcal{M}.$$

This f is called the **Radon–Nikodym derivative**: $f = \dfrac{\mathrm{d}\nu}{\mathrm{d}\mu}$.

The Radon–Nikodym theorem is the measure-theoretic foundation of probability theory. The condition $\nu \ll \mu$ — absolute continuity — is precisely what guarantees that ν admits a density with respect to μ; when μ is Lebesgue measure, the derivative $f = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ is the familiar probability density function. More broadly, the theorem gives rigorous content to the intuition that one measure can be "weighted" relative to another. Readers with a probability background will recognize a special case: given a probability space and a sub-σ algebra, the conditional expectation $\mathbb{E}[Y \mid \mathcal{F}]$ is itself a Radon-Nikodym derivative of an appropriate signed measure with respect to $\mu|_{\mathcal{F}}$

6 Riesz Representation for Hilbert Spaces

Theorem 6.1: Riesz Representation Theorem for H

Let H be a Hilbert space and $\ell : H \to \mathbb{C}$ a bounded linear functional. Then there exists a unique $w \in H$ such that

$$\ell(v) = \langle v, w \rangle \quad \text{for all } v \in H.$$

In other words, every continuous linear functional on H is represented by an inner product with a fixed vector: $H^* \cong H$.

This theorem is the key analytic input for proving the Radon–Nikodym theorem.

[1]Proof via the Riesz Representation Theorem for Hilbert spaces: define a bounded linear functional ℓ on $L^2(X, \mathrm{d}(\mu + |\nu|))$ by $\ell(\psi) = \int \psi \, \mathrm{d}\nu$; by Riesz, $\ell(\psi) = \langle \psi, g \rangle$ for some $g \in L^2$, and $f = g/(1-g)\chi_{\{g<1\}}$ gives the Radon–Nikodym derivative. See proof section.

Proofs of Key Results

Theorem 6.1: Riesz Representation for Hilbert Spaces

We first establish the orthogonal decomposition lemma.

Proposition (Orthogonal Decomposition). *For any closed subspace $S \subset H$, there is a decomposition $H = S \oplus S^\perp$, where $S^\perp := \{w \in H : \langle v, w\rangle = 0\ \forall v \in S\}$.*

Proof. Given $u \in H$, set $d := d(u,S) = \inf\{\|u - v\| : v \in S\}$. Choose a sequence $\{v_n\} \subset S$ with $\|u - v_n\| \to d$. By the **parallelogram law**:

$$\|v_n - v_m\|^2 = 2\|u - v_n\|^2 + 2\|u - v_m\|^2 - 4\left\|u - \frac{v_n + v_m}{2}\right\|^2.$$

Since $(v_n + v_m)/2 \in S$, we have $\|u - (v_n + v_m)/2\| \geq d$, so as $n, m \to \infty$:

$$\limsup_{n,m\to\infty} \|v_n - v_m\|^2 \leq 2d^2 + 2d^2 - 4d^2 = 0.$$

Thus $\{v_n\}$ is Cauchy. Since S is closed, the limit $v \in S$ exists. Define $w := u - v$.

We claim $w \in S^\perp$. For any $s \in S$, the minimality of v gives, by the first-order condition $\frac{\mathrm{d}}{\mathrm{d}t}\|u - (v + ts)\|^2\big|_{t=0} = 0$:

$$\langle w, s\rangle = 0 \quad \forall s \in S.$$

Hence $w \in S^\perp$ and $u = v + w \in S \oplus S^\perp$. □

Proof of Riesz Representation. Let $\ell : H \to \mathbb{C}$ be a continuous linear functional. Define $S := \ker(\ell)$. Linearity of ℓ makes S a subspace; continuity makes S closed (preimage of $\{0\}$ under a continuous map). By the Proposition, $H = S \oplus S^\perp$.

If $S^\perp = \{0\}$, then $\ell \equiv 0$ and $w = 0$ works. Otherwise, fix $w_1 \in S^\perp$ with $\|w_1\| = 1$. For any $v \in H$, note that

$$u := v\,\ell(w_1) - w_1\,\ell(v)$$

satisfies $\ell(u) = \ell(v)\ell(w_1) - \ell(w_1)\ell(v) = 0$, so $u \in S$. Since $w_1 \in S^\perp$ and $u \in S$, we have $\langle u, w_1\rangle = 0$:

$$0 = \langle v\,\ell(w_1) - w_1\,\ell(v),\ w_1\rangle = \ell(w_1)\langle v, w_1\rangle - \ell(v)\|w_1\|^2.$$

Since $\|w_1\| = 1$, this gives $\ell(v) = \ell(w_1)\langle v, w_1\rangle = \langle v, \overline{\ell(w_1)}\,w_1\rangle$. Setting $w := \overline{\ell(w_1)}\,w_1 \in S^\perp$ gives $\ell(v) = \langle v, w\rangle$ for all $v \in H$.

Uniqueness. If w' also satisfies $\ell(v) = \langle v, w'\rangle$ for all v, then $\langle v, w - w'\rangle = 0$ for all v. Taking $v = w - w'$: $\|w - w'\|^2 = 0$, so $w = w'$. □

Theorem 5.1: Radon–Nikodym Theorem

Proof. **Setup.** Let $\rho := \mu + \nu$. Since μ and ν are σ-finite, so is ρ. Define $\ell : L^2(X, \mathrm{d}\rho) \to \mathbb{C}$ by

$$\ell(\psi) := \int_X \psi \, \mathrm{d}\nu.$$

Since $\nu \leq \rho$ (as $\rho = \mu + \nu$ and $\mu \geq 0$), by Cauchy–Schwarz applied to $L^2(X, \mathrm{d}\rho)$:

$$|\ell(\psi)| = \left| \int_X \psi \, \mathrm{d}\nu \right| \leq \int_X |\psi| \, \mathrm{d}\nu \leq \int_X |\psi| \, \mathrm{d}\rho = \langle |\psi|, 1 \rangle_{L^1(X,\mathrm{d}\rho)} \leq \|\psi\|_{L^2(X,\mathrm{d}\rho)} \rho(X)^{1/2},$$

so ℓ is bounded on $L^2(X, \mathrm{d}\rho)$ (using σ-finiteness to localise to finite-measure pieces).

Apply Riesz. By the Riesz Representation Theorem, there exists $g \in L^2(X, \mathrm{d}\rho)$ such that

$$\ell(\psi) = \langle \psi, g \rangle_{L^2(X,\mathrm{d}\rho)} = \int_X \psi \, g \, \mathrm{d}\rho \quad \forall \psi \in L^2.$$

Since ν is a real-valued measure, ℓ maps to $\mathbb{R}$, so g is real-valued ρ-a.e. For any $E \in \mathcal{M}$: $\nu(E) = \ell(\chi_E) = \int_E g \, \mathrm{d}\rho \geq 0$ (since $\nu \geq 0$), so $g \geq 0$ ρ-a.e. Also $\nu(E) \leq \rho(E) = \int_E \mathrm{d}\rho$ for all E gives $g \leq 1$ ρ-a.e. Thus $g : X \to [0, 1]$ ρ-a.e.

Key identity. For any $\psi \in L^2(X, \mathrm{d}\rho)$, expanding $\mathrm{d}\rho = \mathrm{d}\mu + \mathrm{d}\nu$:

$$\int_X \psi \, \mathrm{d}\nu = \int_X \psi g \, \mathrm{d}\rho = \int_X \psi g \, \mathrm{d}\mu + \int_X \psi g \, \mathrm{d}\nu,$$

which rearranges to:

$$\int_X \psi(1 - g) \, \mathrm{d}\nu = \int_X \psi g \, \mathrm{d}\mu. \qquad (*)$$

Decomposition. Define $A := \{g < 1\}$ and $B := \{g = 1\}$, and set

$$\nu_a(E) := \nu(A \cap E), \qquad \nu_s(E) := \nu(B \cap E).$$

Then $\nu = \nu_a + \nu_s$ (since $A \sqcup B = X$ up to a null set).

Singularity: $\nu_s \perp \mu$. Apply $(*)$ with $\psi = \chi_B$. On B, $g = 1$ so $1 - g = 0$:

$$0 = \int_X \chi_B(1 - g) \, \mathrm{d}\nu = \int_X \chi_B g \, \mathrm{d}\mu = \int_B 1 \, \mathrm{d}\mu = \mu(B).$$

Hence $\mu(B) = 0$, so μ cannot see B. Since ν_s is supported on B and ν_a on A, we have $\nu_s \perp \mu$.

Absolute continuity: $\nu_a \ll \mu$ with density $f = g/(1 - g)$. Apply $(*)$ with $\psi := \chi_E(1 + g +$

$g^2 + \cdots + g^n)$ for any $E \in \mathcal{M}$:

$$\int_{E \cap A} (1 - g^{n+1}) \, \mathrm{d}\nu = \int_E g(1 + g + \cdots + g^n) \, \mathrm{d}\mu.$$

As $n \to \infty$: on A, $g < 1$ so $g^{n+1} \to 0$. The left side is dominated by $\nu_a(E) < \infty$, so DCT gives $\int_{E \cap A} \mathrm{d}\nu = \int_E \frac{g}{1-g} \, \mathrm{d}\mu$. That is,

$$\nu_a(E) = \int_E f \, \mathrm{d}\mu, \qquad f := \frac{g}{1-g} \geq 0.$$

This gives $\mathrm{d}\nu_a = f \, \mathrm{d}\mu$, confirming $\nu_a \ll \mu$ with Radon–Nikodym derivative $\dfrac{\mathrm{d}\nu}{\mathrm{d}\mu} = f$. □