Real Analysis II: Study Note

Chapter 8: Abstract Measure Theory

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1 Examples of Measures

This chapter synthesizes the course: every construction from Chapters 1–7 generalizes from Lebesgue measure on $\mathbb{R}^d$ to abstract measure spaces $(X, \mathcal{F}, \mu)$. The key tools are the Carathéodory Extension Theorem (from premeasure to measure) and the Lebesgue–Stieltjes construction.

Roadmap: examples of measures → Carathéodory revisit → Borel regularity → premeasures & Lebesgue–Stieltjes → Carathéodory extension → abstract integration theory → L^p spaces.

Four canonical examples of measures.

(1) **Lebesgue measure:** $(\mathbb{R}^d, \mathcal{M}, m)$ where $\mathcal{M}$ is the Lebesgue σ-algebra. The prototype. (When restricted to the Borel σ-algebra $\mathcal{B} \subsetneq \mathcal{M}$, this is the Borel measure.)

(2) **Weighted measure:** Given measure μ and $f \geq 0$ measurable, define

$$\nu(A) := \int_A f \, \mathrm{d}\mu, \qquad (\mathrm{d}\nu = f \, \mathrm{d}\mu).$$

By MCT, ν is countably additive, hence a measure (the Radon–Nikodym derivative of ν w.r.t. μ is f).

(3) **Pushforward measure:** Given measurable $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ and measure μ on $(X, \mathcal{F})$, define a measure $f_*\mu$ on $(Y, \mathcal{G})$ by

$$(f_*\mu)(A) := \mu(f^{-1}(A)), \qquad A \in \mathcal{G}.$$

Measurability of f ensures $f^{-1}(A) \in \mathcal{F}$ for all $A \in \mathcal{G}$. This is the distribution of f under μ (fundamental in probability: if $(X, \mathcal{F}, \P)$ is a probability space and $f = X$ a random variable, then $f_*\P$ is the law of X).

(4) **Positive functional / Riesz–Markov–Kakutani:** Let X be a locally compact Hausdorff space, $C_c(X)$ the space of continuous compactly-supported functions, and $\ell : C_c(X) \to \mathbb{R}$ a positive linear functional ($f \geq 0 \Rightarrow \ell(f) \geq 0$). Then there exists a unique **Radon measure** μ on X (locally finite, inner regular Borel measure) such that

$$\ell(f) = \int f \, \mathrm{d}\mu \quad \forall f \in C_c(X).$$

Every positive functional on $C_c(X)$ is integration against a unique Radon measure.

2 Revisiting Carathéodory's Theorem

Definition 2.1: Outer Measure (General)

$\mu^* : \mathcal{P}(X) \to [0, \infty]$ is an **outer measure** if: $\mu^*(\varnothing) = 0$; $E_1 \subset E_2 \Rightarrow \mu^*(E_1) \leq \mu^*(E_2)$; and $\mu^*(\bigcup_j E_j) \leq \sum_j \mu^*(E_j)$ (countable sub-additivity).

To extract a measure from μ^*, define the **Carathéodory σ-algebra**:

$$\mathcal{M} := \left\{ E \subset X : \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c) \ \forall A \subset X \right\}.$$

$E \in \mathcal{M}$ means E can be used to partition any set A into two pieces that μ^* adds exactly. This is the Carathéodory measurability condition from Chapter 1, revisited in full generality.

Theorem 2.2: Carathéodory's Theorem

$\mathcal{M}$ is a σ-algebra and $\mu := \mu^*|_{\mathcal{M}}$ is a complete measure: all subsets of μ^*-null sets belong to $\mathcal{M}$, making $(X, \mathcal{M}, \mu)$ a complete measure space. In particular, $\mathcal{M}$ contains all sets of μ^*-measure zero.

3 Metric Outer Measures and Borel Regularity

Definition 3.1: Metric Outer Measure

On a metric space (X, d), an outer measure μ^* is a **metric outer measure** (MOM) if

$$d(A, B) > 0 \ \Rightarrow \ \mu^*(A \cup B) = \mu^*(A) + \mu^*(B).$$

Theorem 3.2

If μ^* is a metric outer measure and $\mathcal{M}$ is obtained by Carathéodory's theorem, then all Borel sets are in $\mathcal{M}$ (i.e., $\mathcal{B} \subset \mathcal{M}$).

Definition 3.3: Borel Regularity

On a metric space (X, d), a **Borel measure** is a measure defined on $\mathcal{B}$ (the Borel σ-algebra, generated by open sets). A ball in (X, d) is $B(x_0, r) := \{x \in X : d(x, x_0) < r\}$.
A Borel measure m on (X, d) is **(inner/outer) regular** if given $E \in \mathcal{B}$ and $\varepsilon > 0$:

- **Outer regularity:** $\exists$ open $U \supset E$ with $m(U \setminus E) < \varepsilon$.
- **Inner regularity:** $\exists$ closed $F \subset E$ with $m(E \setminus F) < \varepsilon$.

Theorem 3.4: Borel Regularity on Metric Spaces

Let m be a locally finite Borel measure on a metric space (X,d) (i.e., every point has a neighbourhood of finite m-measure). Then m is outer regular: for every $E \in \mathcal{B}$ and $\varepsilon > 0$, there exists open $U \supset E$ with $m(U \setminus E) < \varepsilon$. If moreover (X,d) is separable, then m is also inner regular.

4 Premeasures and the Carathéodory Extension Theorem

Definition 4.1: Algebra and Premeasure

An **algebra** $\mathcal{A}$ on X is closed under complements and finite unions/intersections (not necessarily countable). A **premeasure** on $\mathcal{A}$ is $\mu_0 : \mathcal{A} \to [0,\infty]$ with $\mu_0(\varnothing) = 0$ and, if $\{E_i\} \subset \mathcal{A}$ are disjoint with $\bigsqcup_i E_i \in \mathcal{A}$:

$$\mu_0\left(\bigsqcup_i E_i\right) = \sum_i \mu_0(E_i).$$

(Countable additivity is required only when the union stays in $\mathcal{A}$; a σ-algebra would always keep it in, but $\mathcal{A}$ need not.)

Main example: Lebesgue–Stieltjes measure. Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right-continuous[1]. Define the algebra $\mathcal{A}$ generated by half-open intervals $(a,b]$, and set

$$\mu_0\big((a,b]\big) := F(b) - F(a).$$

Then μ_0 is a premeasure on $\mathcal{A}$. When $F(x) = x$ this recovers Lebesgue measure; for a general CDF F this gives a Borel probability measure μ_F with $\mu_F((a,b]) = F(b) - F(a)$.

Why right-continuity? The premeasure must be σ-additive on $\mathcal{A}$. For a decreasing sequence of half-open intervals $(a_n,b_n] \searrow \varnothing$, one needs $\mu_0((a_n,b_n]) = F(b_n) - F(a_n) \to 0$. This fails if F has a left-limit jump at some b: right-continuity $F(b^+) = F(b)$ ensures $F(b_n) - F(a_n) \to 0$ as $(a_n,b_n] \searrow \varnothing$. Left-continuous F would instead naturally generate measures on $[a,b)$ intervals.

Lemma 4.2: Outer Measure from a Premeasure

Given a premeasure μ_0 on $\mathcal{A}$, define

$$\mu^*(A) := \inf\left\{\sum_j \mu_0(R_j) : A \subset \bigcup_j R_j,\ R_j \in \mathcal{A}\right\}.$$

Then μ^* is an outer measure on X consistent with μ_0 on $\mathcal{A}$.

[1] Right continuity might reminds the reader the CDF in probability theory.

Theorem 4.3: Carathéodory Extension Theorem

Given a premeasure μ_0 on an algebra $\mathcal{A}$, there exists a measure μ on a σ-algebra $\mathcal{F}$ with $\mathcal{A} \subset \mathcal{F}$ such that $\mu|_{\mathcal{A}} = \mu_0$. If μ_0 is σ-finite, this extension is unique and $\mathcal{F}$ can be taken to be $\sigma(\mathcal{A})$.

Proof: Apply Lemma 4.2 to get μ^*, then apply Carathéodory's Theorem to get $\mu = \mu^*|_{\mathcal{M}}$.

5 Abstract Integration Theory

All of Chapters 2–3 generalizes verbatim to an abstract measure space $(X, \mathcal{F}, \mu)$.

(1) $\mathcal{F}$ defines **measurable functions** $f : X \to \mathbb{R}$ (preimages of Borel sets are in $\mathcal{F}$).

(2) **Simple functions:** $\varphi = \sum_j c_j \chi_{E_j}$, $E_j \in \mathcal{F}$. Define $\int_X \varphi \, \mathrm{d}\mu := \sum_j c_j \mu(E_j)$.

(3) **Build up:** simple $\to$ bounded $\to$ positive $\to$ integrable, following the exact same steps as Lebesgue.

Egorov's theorem applies in any *finite* measure space (i.e., when $\mu(X) < \infty$). The resulting integral:

$$\int_X f \, \mathrm{d}\mu, \qquad f \text{ is } \mu\text{-integrable if } \int_X |f| \, \mathrm{d}\mu < \infty.$$

Theorem 5.1: Convergence Theorems (Abstract)

The following hold in any measure space $(X, \mathcal{F}, \mu)$:

- **Fatou's Lemma:** If $f_n \geq 0$, then $\displaystyle\int \liminf_n f_n \leq \liminf_n \int f_n$.
- **Monotone Convergence Theorem:** If $0 \leq f_n \nearrow f$ μ-a.e., then $\displaystyle\int f_n \to \int f$.
- **Dominated Convergence Theorem:** If $|f_n| \leq g \in L^1(\mu)$ and $f_n \to f$ μ-a.e., then $\displaystyle\int |f_n - f| \, \mathrm{d}\mu \to 0$.

The proofs are identical to those of Chapter 2, replacing Lebesgue measure with μ throughout.

Example: Lebesgue–Stieltjes integral. Let $F : \mathbb{R} \to \mathbb{R}$ increasing, right-continuous, and μ_F the associated Borel measure. The **Lebesgue–Stieltjes integral** is:

$$\int g \, \mathrm{d}F := \int_{\mathbb{R}} g \, \mathrm{d}\mu_F.$$

If F has a jump of size a_n at point x_n (i.e., $F(x_n) - F(x_n^-) = a_n$), then $\mu_F(\{x_n\}) = a_n$, so:

$$\int g \, \mathrm{d}F = \sum_n g(x_n)\, a_n \quad \text{(if } F \text{ is purely a jump function).}$$

6 L^p Spaces in Abstract Measure Theory

Definition 6.1: Abstract L^p Space

For a measure space $(X, \mathcal{F}, \mu)$ and $1 \leq p \leq \infty$:

$$L^p(X, \mathrm{d}\mu) := \{f : X \to \mathbb{C} \mid f \text{ measurable, } \|f\|_p < \infty\} \big/ \sim$$

where $f \sim g$ iff $f = g$ μ-a.e., $\|f\|_p = \left(\int_X |f|^p \, \mathrm{d}\mu\right)^{1/p}$ for $p < \infty$, and $\|f\|_\infty = \mu\text{-}|f|$.

Theorem 6.2: Minkowski's Inequality

For $p \geq 1$ and $f, g \in L^p(X, \mathrm{d}\mu)$:

$$\|f + g\|_p \;\leq\; \|f\|_p + \|g\|_p.$$

All results from Chapter 3 carry over to $L^p(X, \mathrm{d}\mu)$ for any measure μ:

- **Riesz–Fischer** (completeness of L^p): holds for any measure μ.
- **Hölder's inequality**: holds for any measure μ.
- **Duality** $(L^p)^* \cong L^q$: holds when μ is σ-finite and $1 \leq p < \infty$.
- **Dense subsets** (C_c^∞, simple functions): depend on the structure of (X, μ).